XIV. *On the Mathematical Expression of Observations of Complex Periodical Phenomena; and on Planetary Influence on the Earth's Magnetism.* By CHARLES CHAMBERS, *F.R.S.*, *and* F. CHAMBERS.

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THE writers purpose in the following pages to determine, by BESSEL'S method, a mathematical expression for a periodical phenomenon from observations which are affected by one or more other periodical phenomena, and to find criteria for judging of the extent to which the expression is affected by these other phenomena; also, having found an expression for a period of known approximation to the truth, to find from it the expression for the true period. In the course of these inquiries, certain ambiguities which affect similarly BESSEL'S expression for a single periodical phenomenon and the results here arrived at will be remarked upon; and, finally, the results will be applied to determine the nature of periodic planetary magnetic influence in particular cases.

2. In BESSEL'S paper "On the Determination of the Law of a Periodic Phenomenon" (a translation of which has been published by the Meteorological Committee in the Quarterly Weather Report, part iv. 1870), the author describes, in Section VII., how periodical phenomena which depend on two or more angles can be developed from observations of the same; and he remarks upon the simplicity of a certain class of cases in which both angles are exact measures of 2π, and one is a multiple of the other. In the description of the process occur the following words:—

"If we designate the two angles by x, x', then in the expression

$$y=p+p_1 \cos x+q_1 \sin x+p_2 \cos 2x+q_2 \sin 2x+ \text{ \&c.}$$

the p, p_1, q_1, &c. which occur are not constant, but depend on x'; and as they are periodic functions of x', each of them has an expression of the form

$$a+a_1 \cos x'+b_1 \sin x'+a_2 \cos 2x'+b_2 \sin 2x'+ \text{ \&c.}$$

It is therefore necessary to deduce this development of p, p_1, q, &c. from the observations. If the available series of observations gives the values of y, not only for values of x $(0, z, 2z, \ldots\ldots (n-1)z)$, which are in arithmetical progression and fill up the period, but also for the combination of each of these values of x with n' values of $x'(0, z', 2z', \ldots\ldots (n'-1)z')$, fulfilling the same conditions, the development has no difficulties." After a perfect elucidation of a type of these cases follow remarks upon comparatively difficult cases, which require more cumbrous methods for eliminating the several constants.

* Subsequently revised by the authors.

3. There is a yet simpler case, the importance of which possibly did not press itself on BESSEL'S attention, but which the present writers (having occasion to describe in connexion with actual observations) find it convenient thus to introduce. It is that of two or more combined phenomena, each of which separately recurs after a certain period, which is of different duration for each phenomenon; and the object of this inquiry will be to determine under what circumstances, and with what degree of accuracy, may the coefficients of the expression, according to BESSEL'S form, of each separate phenomenon be found from a series of observed values of the combined phenomena.

4. As the result will be equally applicable to any number of combined phenomena, we will consider the case of only two, whose periods are respectively $\varkappa$ and $\varkappa'$.

Let $\frac{\varkappa}{\varkappa'}=\frac{f}{g}$, f and g being the least integral numbers that will satisfy this condition*; and x being the interval in time (supposed to be constant) between every two consecutive observations, let the series of observations extend over the time $g\varkappa$ or $f\varkappa'$, of which x is a measure, then will the number (r) of observations be $\frac{g\varkappa}{x}$, and the angles corresponding to the time x under the respective periods $\frac{2\pi}{\varkappa}x=z$ (say) and $\frac{2\pi}{\varkappa'}x=\frac{f}{g}z$; further, the angles corresponding to the time rx or $g\varkappa$ will be $2g\pi$ and $2f\pi$ respectively. If α_m represent the observed value of the combined phenomena at the time mx, and β_m and γ_m be the separate phenomena of which it is composed, β_m recurring after the period $\varkappa$, and γ_m after the period $\varkappa'$, we shall have

$$\beta_m=p_0+p_1\cos mz+q_1\sin mz+p_2\cos 2mz+q_2\sin 2mz+\ \&c., \quad \ldots \quad (1)$$

$$\gamma_m=\mathrm{P}_0+\mathrm{P}_1\cos\frac{f}{g}mz+\mathrm{Q}_1\sin\frac{f}{g}mz+\mathrm{P}_2\cos 2\frac{f}{g}mz+\mathrm{Q}_2\sin 2\frac{f}{g}mz+\ \&c., \quad \ldots \quad (2)$$

$$\alpha_m=\beta_m+\gamma_m. \quad \ldots \quad (3)$$

The observations will furnish r equations of the following form:—

$$\alpha_m=\left\{\begin{array}{l}+p_0+p_1\cos mz+q_1\sin mz+p_2\cos 2mz+q_2\sin 2mz+\&c.\\ +\mathrm{P}_0+\mathrm{P}_1\cos\frac{f}{g}mz+\mathrm{Q}_1\sin\frac{f}{g}mz+\mathrm{P}_2\cos 2\frac{f}{g}mz+\mathrm{Q}_2\sin 2\frac{f}{g}mz+\&c.\end{array}\right\} \quad . \quad (4)$$

and the most probable values of p_0, p_1, q_1, &c., P_0, P_1, Q_1, &c. will be those which give a minimum value to

$$\sum_{m=0}^{m=r-1}\left[-\alpha_m=\left\{\begin{array}{l}+p_0+p_1\cos mz+q_1\sin mz+p_2\cos 2mz+q_2\sin 2mz+\&c.\\ +\mathrm{P}_0+\mathrm{P}_1\cos\frac{f}{g}mz+\mathrm{Q}_1\sin\frac{f}{g}mz+\mathrm{P}_2\cos 2\frac{f}{g}mz+\mathrm{Q}_2\sin 2\frac{f}{g}mz+\&c.,\end{array}\right\}\right]^2 . \ (5)$$

the sum being taken between the limits $m=0$ and $m=r-1$ for integral values of m.

This will have such a value when the differential coefficients of it with respect to

* It will be shown later that it will suffice that f and g *very nearly* satisfy this condition; but it is convenient in what immediately follows to regard them as doing so rigidly.

each of the quantities p_0, p_1, q_1, &c., P_0, P_1, Q_1, &c. vanish*; or, dividing out the factor 2, when

$$\left.\begin{aligned}
0&=\sum_{m=0}^{m=r-1}[-\alpha_m+\beta_m+\gamma_m],\\
0&=\sum_{m=0}^{m=r-1}[\cos\ mz(-\alpha_m+\beta_m+\gamma_m)],\\
0&=\sum_{m=0}^{m=r-1}[\sin\ mz(-\alpha_m+\beta_m+\gamma_m)],\\
0&=\sum_{m=0}^{m=r-1}[\cos 2mz(-\alpha_m+\beta_m+\gamma_m)],\\
0&=\sum_{m=0}^{m=r-1}[\sin 2mz(-\alpha_m+\beta_m+\gamma_m)],\\
&\text{\&c.}\qquad\qquad \text{\&c.},\\
0&=\sum_{m=0}^{m=r-1}[\cos tmz(-\alpha_m+\beta_m+\gamma_m)],\\
0&=\sum_{m=0}^{m=r-1}[\sin tmz(-\alpha_m+\beta_m+\gamma_m)],\\
0&=\sum_{m=0}^{m=r-1}[\cos \frac{f}{g} mz(-\alpha_m+\beta_m+\gamma_m)],\\
0&=\sum_{m=0}^{m=r-1}[\sin \frac{f}{g} mz(-\alpha_m+\beta_m+\gamma_m)],\\
0&=\sum_{m=0}^{m=r-1}[\cos 2\frac{f}{g} mz(-\alpha_m+\beta_m+\gamma_m)],\\
0&=\sum_{m=0}^{m=r-1}[\sin 2\frac{f}{g} mz(-\alpha_m+\beta_m+\gamma_m)],\\
&\text{\&c.}\qquad\qquad \text{\&c.},\\
0&=\sum_{m=0}^{m=r-1}[\cos t\frac{f}{g} mz(-\alpha_m+\beta_m+\gamma_m)],\\
0&=\sum_{m=0}^{m=r-1}[\sin t\frac{f}{g} mz(-\alpha_m+\beta_m+\gamma_m)].
\end{aligned}\right\}\quad\ldots\ldots\ldots (6)$$

Representing by s the suffix of a p, q, P, or Q in a type term of $(\beta_m+\gamma_m)$ in each of the equations (6) in turn, and by t the integral numerical factor of the angle in a type of the sine or cosine which multiplies $(-\alpha_m+\beta_m+\gamma_m)$, let us note that

$$\sum_{m=0}^{m=r-1}\cos s\frac{f}{g}mz\cos tmz=\tfrac{1}{2}\left[\frac{\sin\frac{1}{2}rz\left\{s\frac{f}{g}+t\right\}}{\sin\frac{1}{2}z\left\{s\frac{f}{g}+t\right\}}\cos\tfrac{1}{2}(r+1)z\left\{s\frac{f}{g}+t\right\}+\frac{\sin\frac{1}{2}rz\left\{s\frac{f}{g}-t\right\}}{\sin\frac{1}{2}z\left\{s\frac{f}{g}-t\right\}}\cos\tfrac{1}{2}(r+1)z\left\{s\frac{f}{g}-t\right\}\right],\quad (a)$$

* The second differential coefficients being all squares, and therefore positive, there is no ambiguity as to whether equations (6) correspond to a maximum or minimum value of (5).

which, since $rz=2g\pi$, and if $z\left\{s\frac{f}{g}\pm t\right\}$ be not a multiple of 2π

or $(sf\pm tg)$ not a multiple of r,

$$=0. \quad \ldots \quad \text{(b)}$$

Similarly, under the same conditions,

$$\sum_{m=0}^{m=r-1} \sin s\frac{f}{g}mz \sin tmz=0\,; \quad \ldots \quad \text{(c)}$$

and

$$\left.\begin{aligned} &\sum_{m=0}^{m=r-1} \sin s\frac{f}{g}mz \cos tmz=0, \quad \ldots \quad \text{(d)} \\ &\sum_{m=0}^{m=r-1} \cos s\frac{f}{g}mz \sin tmz=0, \quad \ldots \quad \text{(e)} \end{aligned}\right\} \text{invariably.}$$

If, now, we define as follows,

$a_0=$	$+\{$	the sum of all the values of	p_s	for which	s is 0, or	such that	sg	is a multiple of $r\}$
	$+\{$	” ”	P_s	”	s is 0, or	”	sf	” $\}$,
$a_1=$	$+\{$	” ”	p_s	”	s is 1, or	”	$(s\mp 1)g$	” $\}$
	$+\{$	” ”	P_s	”	s is	”	$(sf\mp g)$	” $\}$,
$b_1=$	$+\{$	” ”	q_s	”	s is 1, or	”	$(s-1)g$	” $\}$
	$-\{$	” ”	q_s	”	s is	”	$(s+1)g$	” $\}$
	$+\{$	” ”	Q_s	”	s ”	”	$(sf-g)$	” $\}$
	$-\{$	” ”	Q_s	”	s ”	”	$(sf+g)$	” $\}$,
$a_2=$	$+\{$	” ”	p_s	”	s is 2, or	”	$(s\mp 2)g$	” $\}$
	$+\{$	” ”	P_s	”	s is	”	$(sf\mp 2g)$	” $\}$,
$b_2=$	$+\{$	” ”	q_s	”	s is 2, or	”	$(s-2)g$	” $\}$
	$-\{$	” ”	q_s	”	s is	”	$(s+2)g$	” $\}$
	$+\{$	” ”	Q_s	”	s ”	”	$(sf-2g)$	” $\}$
	$-\{$	” ”	Q_s	”	s ”	”	$(sf+2g)$	” $\}$,
&c.			&c.		&c.		&c.	
$a_t=$	$+\{$	” ”	p_s	”	s is t, or	”	$(s\mp t)g$	” $\}$
	$+\{$	” ”	P_s	”	s is	”	$(sf\mp tg)$	” $\}$,
$b_t=$	$+\{$	” ”	q_s	”	s is t, or	”	$(s-t)g$	” $\}$
	$-\{$	” ”	q_s	”	s is	”	$(s+t)g$	” $\}$
	$+\{$	” ”	Q_s	”	s ”	”	$(sf-tg)$	” $\}$
	$-\{$	” ”	Q_s	”	s ”	”	$(sf+tg)$	” $\}$,
$A_1=$	$+\{$	” ”	P_s	”	s is 1, or	”	$(s\mp 1)f$	” $\}$
	$+\{$	” ”	p_s	”	s is	”	$(sg\mp f)$	” $\}$,
$B_1=$	$+\{$	” ”	Q_s	”	s is 1, or	”	$(s-1)f$	” $\}$
	$-\{$	” ”	Q_s	”	s is	”	$(s+1)f$	” $\}$
	$+\{$	” ”	q_s	”	s ”	”	$(sg-f)$	” $\}$
	$-\{$	” ”	q_s	”	s ”	”	$(sg+f)$	” $\}$,
$A_2=$	$+\{$	” ”	P_s	”	s is 2, or	”	$(s\mp 2)f$	” $\}$
	$+\{$	” ”	p_s	”	s is	”	$(sg\mp 2f)$	” $\}$,

. (7)

$B_2 = \begin{bmatrix} +\{\text{the sum of all the values of } Q_s \text{ for which } s \text{ is } 2, \text{ or such that } (s-2)f \text{ is a multiple of } r\} \\ -\{\quad ,, \quad ,, \quad Q_s \quad ,, \quad s \text{ is} \quad ,, \quad (s+2f) \quad ,, \quad\} \\ +\{\quad ,, \quad ,, \quad q_s \quad ,, \quad s \quad ,, \quad ,, \quad (sg-2f) \quad ,, \quad\} \\ -\{\quad ,, \quad ,, \quad q_s \quad ,, \quad s \quad ,, \quad ,, \quad (sg+2f) \quad ,, \quad\} \end{bmatrix},$

&c. &c. &c. &c.

$A_t = \begin{bmatrix} +\{\quad ,, \quad ,, \quad P_s \quad ,, \quad s \text{ is } t, \text{ or} \quad ,, \quad (s\mp t)f \quad ,, \quad\} \\ +\{\quad ,, \quad ,, \quad p_s \quad ,, \quad s \text{ is} \quad ,, \quad (sg\mp tf) \quad ,, \quad\} \end{bmatrix},$

$B_t = \begin{bmatrix} +\{\quad ,, \quad ,, \quad Q_s \quad ,, \quad s \text{ is } t, \text{ or} \quad ,, \quad (s-t)f \quad ,, \quad\} \\ -\{\quad ,, \quad ,, \quad Q_s \quad ,, \quad s \text{ is} \quad ,, \quad (s+t)f \quad ,, \quad\} \\ +\{\quad ,, \quad ,, \quad q_s \quad ,, \quad s \quad ,, \quad ,, \quad (sg-tf) \quad ,, \quad\} \\ -\{\quad ,, \quad ,, \quad q_s \quad ,, \quad s \quad ,, \quad ,, \quad (sg+tf) \quad ,, \quad\} \end{bmatrix},$

it is easy by means of (a), (b), (c), (d), (e), and other similar formulæ, to convert the equations (6) into

$$0=\sum_{m=0}^{m=r-1}[-\alpha_m+\beta_m+\gamma_m]=\sum_{m=0}^{m=r-1}[-\alpha_m]+ra_0,$$

whence

$$a_0=\frac{1}{r}\sum_{m=0}^{m=r-1}[\alpha_m];$$

$$0=\sum_{m=0}^{m=r-1}[\cos mz(-\alpha_m+\beta_m+\gamma_m)]=\sum_{m=0}^{m=r-1}[\cos mz(-\alpha_m)]+\frac{r}{2}a_1,$$

whence

$$a_1=\frac{2}{r}\sum_{m=0}^{m=r-1}[\alpha_m\cos mz];$$

$$0=\sum_{m=0}^{m=r-1}[\sin mz(-\alpha_m+\beta_m+\gamma_m)]=\sum_{m=0}^{m=r-1}[\sin mz(-\alpha_m)]+\frac{r}{2}b_1,$$

whence

$$b_1=\frac{2}{r}\sum_{m=0}^{m=r-1}[\alpha_m\sin mz];$$

$$0=\sum_{m=0}^{m=r-1}[\cos 2mz(-\alpha_m+\beta_m+\gamma_m)]=\sum_{m=0}^{m=r-1}[\cos 2mz(-\alpha_m)]+\frac{r}{2}a_2,$$

whence

$$a_2=\frac{2}{r}\sum_{m=0}^{m=r-1}[\alpha_m\cos 2mz];$$

$$0=\sum_{m=0}^{m=r-1}[\sin 2mz(-\alpha_m+\beta_m+\gamma_m)]=\sum_{m=0}^{m=r-1}[\sin 2mz(-\alpha_m)]+\frac{r}{2}b_2,$$

whence

$$b_2=\frac{2}{r}\sum_{m=0}^{m=r-1}[\alpha_m\sin 2mz];$$

&c. &c. &c.;

$$0=\sum_{m=0}^{m=r-1}[\cos tmz(-\alpha_m+\beta_m+\gamma_m)]=\sum_{m=0}^{m=r-1}[\cos tmz(-\alpha_m)]+\frac{r}{2}a_t,$$

whence

$$a_t=\frac{2}{r}\sum_{m=0}^{m=r-1}[\alpha_m\cos tmz];$$

$$0=\sum_{m=0}^{m=r-1}\left[\sin tmz(-\alpha_m+\beta_m+\gamma_m)\right]=\sum_{m=0}^{m=r-1}\left[\sin tmz(-\alpha_m)\right]+\frac{r}{2}b_t,$$

whence

$$b_t=\frac{2}{r}\sum_{m=0}^{m=r-1}\left[\alpha_m \sin tmz\right];$$

$$0=\sum_{m=0}^{m=r-1}\left[\cos\frac{f}{g}mz(-a_m+\beta_m+\gamma_m)\right]=\sum_{m=0}^{m=r-1}\left[\cos\frac{f}{g}mz(-\alpha_m)\right]+\frac{r}{2}A_1,$$

whence

$$A_1=\frac{2}{r}\sum_{m=0}^{m=r-1}\left[\alpha_m\cos\frac{f}{g}mz\right];$$

$$0=\sum_{m=0}^{m=r-1}\left[\sin\frac{f}{g}mz(-\alpha_m+\beta_m+\gamma_m)\right]=\sum_{m=0}^{m=r-1}\left[\sin\frac{f}{g}mz(-\alpha_m)\right]+\frac{r}{2}B_1,$$

whence

$$B_1=\frac{2}{r}\sum_{m=0}^{m=r-1}\left[\alpha_m\sin\frac{f}{g}mz\right];$$

$$0=\sum_{m=0}^{m=r-1}\left[\cos 2\frac{f}{g}mz(-\alpha_m+\beta_m+\gamma_m)\right]=\sum_{m=0}^{m=r-1}\left[\cos 2\frac{f}{g}mz(-\alpha_m)\right]+\frac{r}{2}A_2,$$

whence

$$A_2=\frac{2}{r}\sum_{m=0}^{m=r-1}\left[\alpha_m\cos 2\frac{f}{g}mz\right];$$

$$0=\sum_{m=0}^{m=r-1}\left[\sin 2\frac{f}{g}mz(-\alpha_m+\beta_m+\gamma_m)\right]=\sum_{m=0}^{m=r-1}\left[\sin 2\frac{f}{g}mz(-\alpha_m)\right]+\frac{r}{2}B_2,$$

whence

$$B_2=\frac{2}{r}\sum_{m=0}^{m=r-1}\left[\alpha_m\sin 2\frac{f}{g}mz\right];$$

&c. &c. &c.;

$$0=\sum_{m=0}^{m=r-1}\left[\cos t\frac{f}{g}mz(-a_m+\beta_m+\gamma_m)\right]=\sum_{m=0}^{m=r-1}\left[\cos t\frac{f}{g}mz(-\alpha_m)\right]+\frac{r}{2}A_t,$$

whence

$$A_t=\frac{2}{r}\sum_{m=0}^{m=r-1}\left[\alpha_m\cos t\frac{f}{g}mz\right];$$

$$0=\sum_{m=0}^{m=r-1}\left[\sin t\frac{f}{g}mz(-\alpha_m+\beta_m+\gamma_m)\right]=\sum_{m=0}^{m=r-1}\left[\sin t\frac{f}{g}mz(-\alpha_m)\right]+\frac{r}{2}B_t,$$

whence

$$B_t=\frac{2}{r}\sum_{m=0}^{m=r-1}\left[\alpha_m\sin t\frac{f}{g}mz\right]. \quad (8)$$

5. In any special inquiry, having found by (8) the numerical values of a_0, a_1, b_1, a_2, b_2, &c., A_1, B_1, A_2, B_2, &c., we may insert these in the equations (7), which it will now be desirable to consider the significance of. If our object was simply to find two periodical phenomena which would jointly satisfy the r observations, then this could be done with the same degree of closeness in an infinite variety of ways; for we might give to the several terms of the right-hand members of (7) any arbitrary values consistently with their sum being equal to the left-hand member, and so long as the same coefficient is taken of the same value in all the equations (7). But although all the varieties would agree in giving the same value of the combined phenomena *at any one*

of the r times of observation, they would all generally differ as to its value at any time intermediate between any consecutive two of the r observations. In the first of the equations (7), if we were to attribute the whole of a_0 to p_0 or P_0, it would imply that the phenomenon a_0 occurred at all times irrespective of any periodicity; but if we attribute it all to (say) $p\frac{r}{g}$, it would imply that the phenomenon a_0 occurred only at the times of observation, whilst at intermediate times the corresponding phenomenon would be represented by $a_0 \cos\frac{r}{g} mz$, which passes through a complete cycle of change during the interval between every two consecutive observations, or whilst m passes from one integral value to the next; and combined with this there may be a phenomenon represented by $q\frac{r}{g} \sin\frac{r}{g} mz$ of any arbitrary range. Similarly, the distinction between the different terms of the other equations is that they go through a full cycle of change in different periods; and graphically each term would be represented by a complete wave whose length corresponded to the period of that term.

6. As the mathematical theory of this process affords no criterion for selection, we ought to find reasons apart from it for preferring particular appropriations of a_0, a_1, b_1, &c. to the several component parts of their equalities; otherwise it is clear from what has been said that no useful result will be attained. It may be remarked that an ambiguity, similar to the one under consideration, attaches to BESSEL'S treatment of a single periodical phenomenon, the values corresponding to our a_0, a_1, b_1, &c. being given at the foot of page 26, Section III. of BESSEL'S paper. BESSEL remarks that if we compare a mathematical theory of any periodical phenomenon, based on physical principles, with the observations, his expression for the values of the phenomenon is more convenient for the purpose than the observations themselves—the reason of this being that, as the expression given by the mathematical theory is developed in the form in which the observations have been expressed, the two expressions may be compared term by term, or by equal subordinate periods. This is probably the most important use of the method; and as the most striking features of a variation will generally be those of long period, they may be examined apart from the others. The next most important use of this method is probably that which has for its object the elimination of casual irregularities from the observations; but this is served only when the subordinate variations of short period are rejected; and after such rejection, it must always be borne in mind that the remaining expression is incomplete: this does not, however, interfere with the comparison of the subordinate variations retained with other phenomena of nature involving variations of the same subordinate periods; indeed by indicating the periods followed by the subordinate variations which are of largest amount, it suggests a means of distinguishing other phenomena that on examination may be found to be related to the one which is the subject of the observations. The reason assigned by BESSEL for giving preference to the terms of long period, viz. that "the development of the

expression which represents the given values of y will in general only be interesting when it converges so rapidly that only a few of the first terms have appreciable values," had reference doubtless to the incompleteness of the partial expression—this being of no consequence when the rejected part, the absence of which makes the expression incomplete, is of inconsiderable amount. We may, however, be guided as to the validity of this reason by noting well whether the values of a_t, b_t, A_t, B_t, &c. do themselves become inappreciable whilst t is still small.

7. Now in many special inquiries f, g, and r will have such values that $(s\mp t)g$, $(sf\mp tg)$, $(s\mp t)f$, $(sg\mp tf)$, &c. will first become a multiple of r only when s or t has ceased to be small; in which case, following BESSEL, we may neglect as inappreciable all the terms on the right-hand side of equation (7), except p_0 and P_0 in the first equation and the first term of each of the others; we then have

$$
\left.\begin{aligned}
&p_0+P_0=a_0=\frac{1}{r}\sum_{m=0}^{m=r-1}[\alpha_m],\\
&\left.\begin{aligned}
p_1&=a_1=\frac{2}{r}\sum_{m=0}^{m=r-1}[\alpha_m\cos mz],\\
q_1&=b_1=\frac{2}{r}\sum_{m=0}^{m=r-1}[\alpha_m\sin mz],\\
p_2&=a_2=\frac{2}{r}\sum_{m=0}^{m=r-1}[\alpha_m\cos 2mz],\\
q_2&=b_2=\frac{2}{r}\sum_{m=0}^{m=r-1}[\alpha_m\sin 2mz],\\
&\&c.\qquad \&c.\qquad \&c.,\\
p_t&=a_t=\frac{2}{r}\sum_{m=0}^{m=r-1}[\alpha_m\cos tmz],\\
q_t&=b_t=\frac{2}{r}\sum_{m=0}^{m=r-1}[\alpha_m\sin tmz];
\end{aligned}\right\}\text{which are the same values as those that would be found by applying BESSEL's method to the } r \text{ observations, on the supposition that they are unaffected by the phenomenon whose period is } z'.\\
&\left.\begin{aligned}
P_1&=A_1=\frac{2}{r}\sum_{m=0}^{m=r-1}\left[\alpha_m\cos\frac{f}{g}mz\right],\\
Q_1&=B_1=\frac{2}{r}\sum_{m=0}^{m=r-1}\left[\alpha_m\sin\frac{f}{g}mz\right],\\
P_2&=A_2=\frac{2}{r}\sum_{m=0}^{m=r-1}\left[\alpha_m\cos 2\frac{f}{g}mz\right],\\
Q_2&=B_2=\frac{2}{r}\sum_{m=0}^{m=r-1}\left[\alpha_m\sin 2\frac{f}{g}mz\right],\\
&\&c.\qquad \&c.\qquad \&c.,\\
P_t&=A_t=\frac{2}{r}\sum_{m=0}^{m=r-1}\left[\alpha_m\cos t\frac{f}{g}mz\right],\\
Q_t&=B_t=\frac{2}{r}\sum_{m=0}^{m=r-1}\left[\alpha_m\sin t\frac{f}{g}mz\right];
\end{aligned}\right\}\text{which are the same values as those that would be found by applying BESSEL's method to the } r \text{ observations, on the supposition that they are unaffected by the phenomenon whose period is } z.
\end{aligned}\right\}\quad (9)
$$

8. If instead of applying BESSEL'S process at once to each individual observation, we had begun by finding a mean value $\Sigma\frac{\beta_m}{g}$ (as affected by the other phenomenon) of the one phenomenon at a particular phase of its period $\varkappa$, and then proceeded to apply BESSEL'S process to the $\frac{r}{g}$ mean values of this character, we should have arrived at precisely the same results.

We might also have regarded a hypothetical complex phenomenon of period $g\varkappa$ as being produced solely by the recurrence of the phenomena whose periods are $\varkappa$ and $\varkappa'$, and finding by BESSEL'S process from the r observations the coefficients of its expression —from these determining the coefficients of the expressions for the component periodical phenomena; this, too, would have led to the same results.

9. To conclude this section, we draw from what has preceded the following practical rule for deducing from a series of observations of the combined effect of several independent phenomena (observations taken at equal intervals of time) the coefficients of BESSEL'S series for each separate phenomenon:—Find the least integral numbers f, g, h, &c. which are proportional (or nearly so) to the periods $\varkappa$, $\varkappa'$, $\varkappa''$, &c. of the several phenomena, and let v be the least common multiple of those numbers; choose then for treatment observations extending exactly over some multiple of the period $\frac{v}{f}\varkappa$, and note whether any values of p_s or q_s, P_s or Q_s, &c., for which s is small, other than the first terms, enter into the equations (7); if not, proceed to apply BESSEL'S method to determine from the observations the coefficients of the expression of each phenomenon, just as would be done if the observations were unaffected by the other phenomena.

II.

10. It will be useful further to estimate in what degree the phenomenon whose period is $\varkappa'$ affects the values of the constants p_1, q_1, &c., in the expression of the phenomenon whose period is $\varkappa$, when the number (R) of observations is greater than and not a multiple of r. And here, confining our attention to strictly and exclusively periodical phenomena, we must reject the constant term (p_0+P_0) in the expression for the combined phenomena: this is equivalent to substituting for the original observations $\alpha_0, \alpha_1, \alpha_2, \ldots\ldots \alpha_m$ the excesses of them respectively above their mean value $\Sigma\frac{\alpha_m}{R}$. Let $\frac{c\varkappa}{x}=R=\frac{2c\pi}{z}$, c being an integer, and let $c\varkappa=(d+e)f\kappa'$, d being integral and e a proper fraction. If we represent β_m by the general term

$$[p_s\cos smz+q_s\sin smz],$$

and γ_m by the general term

$$\left[P_s\cos s\frac{f}{g}mz+Q_s\sin s\frac{f}{g}mz\right],$$

the first set of expressions of (6) may be put into the following typical form,

$$\left.\begin{aligned}&\sum_{m=0}^{m=r-1}[p_s\cos smz+q_s\sin smz]\cos tmz\\&=\sum_{m=0}^{m=r-1}[\alpha_m\cos tmz]-\sum_{m=0}^{m=r-1}\left[\mathrm{P}_s\cos s\frac{f}{g}mz+\mathrm{Q}_s\sin s\frac{f}{g}mz\right]\cos tmz,\end{aligned}\right\}\quad . \quad (10)$$

which, in the case before us, is

$$\left.\begin{aligned}&\sum_{m=0}^{m=\mathrm{R}-1}[p_s\cos smz+q_s\sin smz]\cos tmz\\&=\sum_{m=0}^{m=\mathrm{R}-1}[\alpha_m\cos tmz]-\sum_{m=0}^{m=\mathrm{R}-1}\left[\mathrm{P}_s\cos s\frac{f}{g}mz+\mathrm{Q}_s\sin s\frac{f}{g}mz\right]\cos tmz;\end{aligned}\right\}\quad . \quad (11)$$

and since $\mathrm{R}z=2c\pi$, and neglecting, with BESSEL, the terms of β_m and γ_m for which s is not small, and also dividing through by $\frac{\mathrm{R}}{2}$, this becomes

$$p_t=\frac{2}{\mathrm{R}}\sum_{m=0}^{m=\mathrm{R}-1}[\alpha_m\cos tmz]-\frac{2}{\mathrm{R}}\sum_{m=0}^{m=\mathrm{R}-1}\left[\mathrm{P}_s\cos s\frac{f}{g}mz+\mathrm{Q}_s\sin s\frac{f}{g}mz\right]\cos tmz.\quad . \quad (12)$$

But after each successive period fx' the quantity

$$\left[\mathrm{P}_s\cos s\frac{f}{g}mz+\mathrm{Q}_s\sin s\frac{f}{g}mz\right]\cos tmz$$

passes again, in each of its terms, through the same identical values; it is therefore a proper periodical function, and passes at the same phase of each period fx' through some maximum value, which cannot ever be of magnitude so great as the sum of all the P's and Q's disregarding their signs; much less can

$$\frac{1}{\mathrm{R}}\sum_{m=0}^{m=\mathrm{R}-1}\left[\mathrm{P}_s\cos s\frac{f}{g}mz+\mathrm{Q}_s\sin s\frac{f}{g}mz\right]\cos tmz$$

ever reach that sum; hence the last term of (12) can never be so great as twice the sum of all the P's and Q's regardless of signs. Suppose this to be its value at some time during the first period fx', then at no time in the second period fx' can it exceed the half of this, since R will have been at least doubled, whilst the part under the sign of summation cannot have increased; similarly, at no time during the nth period fx' can its value be of greater magnitude than $\frac{2}{n}$ths of the sum of its P's and Q's regardless of signs. Hence if n be made large enough, *i. e.* if the observations be sufficiently extended, this quantity can always be reduced till its effect upon the value of p_t is inappreciable.

11. Now it has been shown in the preceding investigation that, as R increases and passes successively through the values $\frac{fx'}{x}$, $\frac{2fx'}{x}$, $\frac{3fx'}{x}$ . . . $\frac{dfx'}{x}$, &c., the quantity

$$\sum_{m=0}^{m=\mathrm{R}-1}\left[\mathrm{P}_s\cos s\frac{f}{g}mz+\mathrm{Q}_s\sin s\frac{f}{g}mz\right]\cos tmz$$

vanishes at each passage; when, therefore, the series of observations is not sufficiently extensive to obliterate the effect of the last term of (12), it may be worth while, in the first place, to calculate approximately the values of P_1, Q_1, P_2, Q_2, &c., choosing for the purpose a number of observations R′ which very nearly completes an integral number of periods $f\varkappa'$, and thence the value of

$$\Sigma\left[P_s \cos s\frac{f}{g}mz + Q_s \sin s\frac{f}{g}mz\right] \cos tmz$$

for the fractional part of a period $f\varkappa'$ which is in excess of the last completed period.

Similar reasoning, with a similar result, may be applied to each of the expressions of (6), of which (10) is a type.

III.

12. The variations in a series of n observations (equidistant in time) are by hypothesis due to a periodical phenomenon whose true expression is

$$\alpha_m = p_0 + p_1 \cos mz + q_1 \sin mz + p_2 \cos 2mz + q_2 \sin 2mz + \&c., \quad . \quad . \quad . \quad (13)$$

in relation to which $z=\frac{2c\pi}{x}$, c=a constant integer not small, $\varkappa$=the period of the phenomenon, $x=\varkappa\frac{c}{n}$=the interval of time corresponding to the angle z, t=the time reckoned from the commencement of the observations, and $m=\frac{t}{x}$. Let the interval between successive observations be $(x+\Delta x)$, so that the n observations will extend over a period $n(x+\Delta x)=c(\varkappa+\Delta\varkappa)$. The angle $(z+i)$ which corresponds to the interval of time $(x+\Delta x)$ will be equal to $z\frac{x+\Delta x}{x}=z+z\frac{\Delta x}{x}$ or $i=z\frac{\Delta x}{x}=z\frac{\Delta\varkappa}{\varkappa}$; let this be so small that smi is also a small angle, s being the suffix of a p or q. Under these conditions, to find the coefficients p_1, q_1, &c. Let it first be observed that the condition that smi is a small angle, implies that n has been so chosen that $(\varkappa+\Delta\varkappa)$ approximates as closely as possible to the known or assumed value of $\varkappa$. The phenomenon α_m occurring at the time mx, let that which occurs at the time $m(x+\Delta x)$ be called $\alpha_{m'}$; then we shall have

$$\alpha_{m'} = p_0 + p_1 \cos m(z+i) + q_1 \sin m(z+i) + p_2 \cos 2m(z+i) + q_2 \sin 2m(z+i) + \&c., \quad (14)$$

the general term of which is $\{p_s \cos sm(z+i) + q_s \sin sm(z+i)\}$, where s represents the positive integral suffix of a p or q; and we may, for shortness, write

$$\alpha_{m'} = [p_s \cos sm(z+i) + q_s \sin sm(z+i)], \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

the square brackets indicating that the general term within them is to represent the sum of its series of values when for s is put 0, 1, 2, 3, &c. . . . . successively, whence

$$\alpha_{m'} = [p_s(\cos smz \cos smi - \sin smz \sin smi) + q_s(\sin smz \cos smi + \cos smz \sin smi)]; \quad (16)$$

and smi being a small angle, we may write for its sine smi, and for its cosine $(1-\frac{1}{2}s^2m^2i^2)$, when we obtain

$$\left.\begin{aligned}\alpha_{m'}=\Big[(p_s\cos smz+q_s\sin smz)+si(q_s m\cos smz-p_s m\sin smz)\\ \frac{s^2i^2}{2}(p_s m^2\cos smz+q_s m^2\sin smz)\Big].\end{aligned}\right\}\quad . \quad . \quad . \quad (17)$$

Multiplying both sides by $\cos tmz$, t being any positive integer,

$$\left.\begin{aligned}\alpha_{m'}\cos tmz=\alpha_m\cos tmz+\Big[si(q_s m\cos smz\cos tmz-p_s m\sin smz\cos tmz)\\ -\frac{s^2i^2}{2}(p_s m^2\cos smz\cos tmz+q_s m^2\sin smz\cos tmz)\Big];\end{aligned}\right\}\quad . \quad (18)$$

and taking the sum on both sides from $m=0$ to $m=(n-1)$,

$$\left.\begin{aligned}&\sum_{m=0}^{m=n-1}\alpha_{m'}\cos tmz=\sum_{m=0}^{m=n-1}\alpha_m\cos tmz\\ &+\sum_{m=0}^{m=n-1}\Big[\frac{si}{2}\{q_s(m\cos(s+t)mz+m\cos(s-t)mz)-p_s(m\sin(s+t)mz+m\sin(s-t)mz)\}\\ &\quad-\frac{s^2i^2}{4}\{p_s(m^2\cos(s+t)mz+m^2\cos(s-t)mz)+q_s(m^2\sin(s+t)mz+m^2\sin(s-t)mz)\}\Big].\end{aligned}\right\}\quad (19)$$

Now observing, from the collected equations at the end of the first set of demonstrations in the Appendix, that when $nv=2c\pi$, and according as v is not or is 0 or a multiple of 2π,

$$\left.\begin{aligned}&\sum_{m=0}^{m=n-1} m\cos mv=-\frac{n}{2}, &&\text{or } \frac{n^2}{2}-\frac{n}{2},\\ &\sum_{m=0}^{m=n-1} m\sin mv=-\frac{n}{2}\cot\frac{v}{2}, &&\text{or } 0,\\ &\sum_{m=0}^{m=n-1} m^2\cos mv=-\frac{n^2}{2}+\frac{n}{2}\frac{1}{\sin^2\frac{v}{2}}, &&\text{or } \frac{n^3}{3}-\frac{n^2}{2}+\frac{n}{6},\\ &\sum_{m=0}^{m=n-1} m^2\sin mv=-\frac{n^2}{2}\cot\frac{v}{2}, &&\text{or } 0.\end{aligned}\right\}\quad . \quad . \quad . \quad . \quad . \quad (20)$$

And as in equation (19) $nz=2c\pi$, applying equations (20), equation (19) becomes

$$\begin{aligned}\sum_{m=0}^{m=n-1}\alpha_{m'}\cos tmz=\sum_{m=0}^{m=n-1}\alpha_m\cos tmz+\Big[\frac{si}{2}\Big\{q_s\Big(-\frac{n}{2}-\frac{n}{2}\Big)-p_s\Big(-\frac{n}{2}\cot\frac{s+t}{2}z-\frac{n}{2}\cot\frac{s-t}{2}z\Big)\Big\}\\ -\frac{s^2i^2}{4}\Big\{p_s\Big(-\frac{n^2}{2}+\frac{n}{2}\frac{1}{\sin^2\frac{s+t}{2}z}-\frac{n^2}{2}+\frac{n}{2}\frac{1}{\sin^2\frac{s-t}{2}z}\Big)\\ +q_s\Big(-\frac{n^2}{2}\cot\frac{s+t}{2}z-\frac{n^2}{2}\cot\frac{s-t}{2}z\Big)\Big\}\Big],\end{aligned}$$

$$\left.\begin{aligned}
&\text{or } =\sum_{m=0}^{m=n-1}\alpha_m\cos tmz+\left[\frac{si}{2}\left\{q_s\left(-\frac{n}{2}+\frac{n^2}{2}-\frac{n}{2}\right)-p_s\left(-\frac{n}{2}\cot\frac{s+t}{2}z+0\right)\right\}\right.\\
&\qquad -\frac{s^2i^2}{4}\left\{p_s\left(-\frac{n^2}{2}+\frac{n}{2}\frac{1}{\sin^2\frac{s+t}{2}z}+\frac{n^3}{3}-\frac{n^2}{2}+\frac{n}{6}\right)\right.\\
&\qquad \left.\left.+q_s\left(-\frac{n^2}{2}\cot\frac{s+t}{2}z+0\right)\right\}\right],\\
&\text{or } =\sum_{m=0}^{m=n-1}\alpha_m\cos tmz+\left[\frac{si}{2}\left\{q_s\left(\frac{n^2}{2}-\frac{n}{2}-\frac{n}{2}\right)-p_s\left(0-\frac{n}{2}\cot\frac{s-t}{2}z\right)\right\}\right.\\
&\qquad -\frac{s^2i^2}{4}\left\{p_s\left(\frac{n^3}{3}-\frac{n^2}{2}+\frac{n}{6}-\frac{n^2}{2}+\frac{n}{2}\frac{1}{\sin^2\frac{s-t}{2}z}\right)\right.\\
&\qquad \left.\left.+q_s\left(0-\frac{n^2}{2}\cot\frac{s-t}{2}z\right)\right\}\right],\\
&\text{or } =\sum_{m=0}^{m=n-1}\alpha_m\cos tmz+\left[\frac{si}{2}\{q_sn(n-1)\}-\frac{s^2i^2}{4}\left\{p_s\left(\frac{2}{3}n^3-n^2+\frac{n}{3}\right)\right\}\right],
\end{aligned}\right\}\quad(21)$$

according as neither $(s+t)z$ nor $(s-t)z$ is 0 or a multiple of 2π; as $(s+t)z$ is not, and $(s-t)z$ is 0 or a multiple of 2π; as $(s+t)z$ is, and $(s-t)z$ is not 0 or a multiple of 2π; or as both $(s+t)z$ and $(s-t)z$ are multiples of 2π.

And multiplying both sides by $\frac{2}{n}$,

$$\left.\begin{aligned}
&\frac{2}{n}\sum_{m=0}^{m=n-1}\alpha_{m'}\cos tmz=\frac{2}{n}\sum_{m=0}^{m=n-1}\alpha_m\cos tmz+\left[\frac{si}{2}\left\{-2q_s+p_s\left(\cot\frac{s+t}{2}z+\cot\frac{s-t}{2}z\right)\right\}\right.\\
&\qquad \left.-\frac{s^2i^2}{4}\left\{p_s\left(-2n+\frac{1}{\sin^2\frac{s+t}{2}z}+\frac{1}{\sin^2\frac{s-t}{2}z}\right)-q_s\left(n\cot\frac{s+t}{2}z+n\cot\frac{s-t}{2}z\right)\right\}\right],\\
&\qquad\text{or}=\frac{2}{n}\sum_{m=0}^{m=n-1}\alpha_m\cos tmz+\left[\frac{si}{2}\left\{q_s(n-2)+p_s\left(\cot\frac{s+t}{2}z\right)\right\}\right.\\
&\qquad \left.-\frac{s^2i^2}{4}\left\{p_s\left(\tfrac{2}{3}n^2-2n+\tfrac{1}{3}+\frac{1}{\sin^2\frac{s+t}{2}z}\right)-q_s\left(n\cot\frac{s+t}{2}z\right)\right\}\right],\\
&\qquad\text{or}=\frac{2}{n}\sum_{m=0}^{m=n-1}\alpha_m\cos tmz+\left[\frac{si}{2}\left\{q_s(n-2)+p_s\left(\cot\frac{s-t}{2}z\right)\right\}\right.\\
&\qquad \left.-\frac{s^2i^2}{4}\left\{p_s\left(\tfrac{2}{3}n^2-2n+\tfrac{1}{3}+\frac{1}{\sin^2\frac{s-t}{2}z}\right)-q_s\left(n\cot\frac{s-t}{2}z\right)\right\}\right],\\
&\qquad\text{or}=\frac{2}{n}\sum_{m=0}^{m=n-1}\alpha_m\cos tmz+\left[\frac{si}{2}\left\{q_s\,2(n-1)\right\}-\frac{s^2i^2}{4}\{p_s(\tfrac{4}{3}n^2-2n+\tfrac{2}{3})\}\right],
\end{aligned}\right\}\quad(22)$$

according as neither $(s+t)z$ nor $(s-t)z$ is 0 or a multiple of 2π; as $(s+t)z$ is not, but $(s-t)z$ is 0 or a multiple of 2π; as $(s+t)z$ is, but $(s-t)z$ is not 0 or a multiple of 2π; or as both $(s+t)z$ and $(s-t)z$ are multiples of 2π.

Now by BESSEL'S process, and assuming, as we shall, that only a few of the first terms of the expression for α_m have considerable coefficients,

$$\frac{2}{n}\sum_{m=0}^{m=n-1}\alpha_m\cos tmz=p_t,\quad\text{and}\quad\frac{2}{n}\sum_{m=0}^{m=n-1}\alpha_m\sin tmz=q_t.$$

Also let

$$\frac{2}{n}\sum_{m=0}^{m=n-1}\alpha_{m'}\cos tmz=\mathrm{P}_t,\quad\text{and}\quad\frac{2}{n}\sum_{m=0}^{m=n-1}\alpha_{m'}\sin tmz=\mathrm{Q}_t.$$

Therefore, writing a_t and a_t respectively for the coefficients of i and i^2 in (22), and transposing,

$$p_t=\mathrm{P}_t-a_ti-\mathrm{a}_ti^2. \qquad (23)$$

Proceeding in a similar manner, we find :—

$$\left.\begin{aligned}&\alpha_{m'}\sin tmz=\alpha_m\sin tmz\\&+\Big[si(q_sm\cos smz\sin tmz-p_sm\sin smz\sin tmz)-\frac{s^2i^2}{2}(p_sm^2\cos smz\sin tmz\\&+q_sm^2\sin smz\sin tmz)\Big];\end{aligned}\right\}\qquad(24)$$

$$\left.\begin{aligned}&\sum_{m=0}^{m=n-1}\alpha_{m'}\sin tmz=\sum_{m=0}^{m=n-1}\alpha_m\sin tmz\\&+\sum_{m=0}^{m=n-1}\Big[\frac{si}{2}\Big\{q_s\big(m\sin(s+t)mz-m\sin(s-t)mz\big)-p_s\big(m\cos(s-t)mz-m\cos(s+t)mz\big)\Big\}\\&-\frac{s^2i^2}{4}\Big\{p_s\big(m^2\sin(s+t)mz-m^2\sin(s-t)mz\big)+q_s\big(m^2\cos(s-t)mz-m^2\cos(s+t)mz\big)\Big\}\Big];\end{aligned}\right\}\qquad(25)$$

$$\left.\begin{aligned}&\sum_{m=0}^{m=n-1}\alpha_{m'}\sin tmz=\sum_{m=0}^{m=n-1}\alpha_m\sin tmz\\&+\Big[\frac{si}{2}\Big\{q_s\Big(-\frac{n}{2}\cot\frac{s+t}{2}z+\frac{n}{2}\cot\frac{s-t}{2}z\Big)-p_s\Big(-\frac{n}{2}+\frac{n}{2}\Big)\Big\}-\frac{s^2i^2}{4}\Big\{p_s\Big(-\frac{n^2}{2}\cot\frac{s+t}{2}z\\&+\frac{n^2}{2}\cot\frac{s-t}{2}z\Big)+q_s\Big(-\frac{n^2}{2}+\frac{n}{2}\frac{1}{\sin^2\frac{s-t}{2}z}+\frac{n^2}{2}-\frac{n}{2}\frac{1}{\sin^2\frac{s+t}{2}z}\Big)\Big\}\Big],\\&\qquad\text{or }=\sum_{m=0}^{m=n-1}\alpha_m\sin tmz\\&+\Big[\frac{si}{2}\Big\{q_s\Big(-\frac{n}{2}\cot\frac{s+t}{2}z+0\Big)-p_s\Big(\frac{n^2}{2}-\frac{n}{2}+\frac{n}{2}\Big)\Big\}-\frac{s^2i^2}{4}\Big\{p\Big(-\frac{n^2}{2}\cot\frac{s+t}{2}z+0\\&+q_s\Big(\frac{n^3}{3}-\frac{n^2}{2}+\frac{n}{6}+\frac{n^2}{2}-\frac{n}{2}\frac{1}{\sin^2\frac{s+t}{2}z}\Big)\Big\}\Big],\end{aligned}\right\}\qquad(26)$$

$$\text{or} = \sum_{m=0}^{m=n-1} \alpha_m \sin tmz$$

$$+\left[\frac{si}{2}\left\{q_s\left(0+\frac{n}{2}\cot\frac{s-t}{2}z\right)-p_s\left(-\frac{n}{2}-\frac{n^2}{2}+\frac{n}{2}\right)\right\}-\frac{s^2i^2}{4}\left\{p_s\left(0+\frac{n^2}{2}\cot\frac{s-}{2}z\right)\right.\right.$$

$$\left.\left.+q_s\left(-\frac{n^2}{2}+\frac{n}{2}\frac{1}{\sin^2\frac{s-t}{2}z}-\frac{n^3}{3}+\frac{n^2}{2}-\frac{n}{6}\right)\right\}\right],$$

$$\text{or} = \sum_{m=0}^{m=n-1} \alpha_m \sin tmz,$$

according as neither $(s+t)z$ nor $(s-t)z$ is 0 or a multiple of 2π; as $(s+t)z$ is not, but $(s-t)z$ is 0 or a multiple of 2π; as $(s+t)z$ is, but $(s-t)z$ is not 0 or a multiple of 2π; or as both $(s+t)z$ and $(s-t)z$ are multiples of 2π.

$$\frac{2}{n}\sum_{m=0}^{m=n-1} \alpha_{m'} \sin tmz = \frac{2}{n}\sum_{m=0}^{m=n-1} \alpha_m \sin tmz$$

$$+\left[\frac{si}{2}\left\{q_s\left(\cot\frac{s-t}{2}z-\cot\frac{s+t}{2}z\right)\right\}-\frac{s^2i^2}{4}\left\{p_s\left(n\cot\frac{s-t}{2}z-n\cot\frac{s+t}{2}z\right)\right.\right.$$

$$\left.\left.+q_s\left(\frac{1}{\sin^2\frac{s-t}{2}z}-\frac{1}{\sin^2\frac{s+t}{2}z}\right)\right\}\right],$$

$$\text{or} = \frac{2}{n}\sum_{m=0}^{m=n-1} \alpha_m \sin tmz$$

$$+\left[\frac{si}{2}\left\{q_s\left(-\cot\frac{s+t}{2}z\right)-p_s n\right\}-\frac{s^2i^2}{4}\left\{p_s\left(-n\cot\frac{s+t}{2}z\right)+q_s\left(\frac{2n^2}{3}+\tfrac{1}{3}-\frac{1}{\sin^2\frac{s+t}{2}z}\right)\right\}\right], \qquad (27)$$

$$\text{or} = \frac{2}{n}\sum_{m=0}^{m=n-1} \alpha_m \sin\alpha \sin tmz$$

$$+\left[\frac{si}{2}\left\{q_s\left(\cot\frac{s-t}{2}z\right)+p_s n\right\}-\frac{s^2i^2}{4}\left\{p_s n\cot\frac{s-t}{2}z+q_s\left(-\tfrac{2}{3}n^2-\tfrac{1}{3}+\frac{1}{\sin^2\frac{s-t}{2}z}\right)\right\}\right],$$

$$\text{or} = \frac{2}{n}\sum_{m=0}^{m=n-1} \alpha_m \sin tmz,$$

according as neither $(s+t)z$ nor $(s-t)z$ is 0 or a multiple of 2π; as $(s+t)z$ is not, but $(s-t)z$ is 0 or a multiple of 2π; as $(s+t)z$ is, but $(s-t)z$ is not 0 or a multiple of 2π; or as both $(s+t)z$ and $(s-t)z$ are multiples of 2π.

And writing b_t and b_t respectively for the coefficients of i and i^2 in (27), and transposing,

$$q_t = \mathrm{Q}_t - b_t i - \mathrm{b}_t i^2 . \qquad (28)$$

13. From the general expressions (23) and (28), for the coefficients p_t, q_t we may now write down the particular values p_1, p_2, p_3, q_1, q_2, q_3 for the particular case in which, whilst neither s nor t is taken above 3, neither $(s-t)z$ nor $(s+t)z$ is ever a multiple of 2π; and at the same operation we may substitute for the general terms in which a_t, a_t, b_t, b_t are expressed, the series of terms obtained by giving s the values, 1, 2, 3 ...... successively, observing also that when $s=0$ these terms vanish. We have then,

$$\left.\begin{aligned}p_1=\mathrm{P}_1&-\frac{i}{2}\left\{q_1(n-2)+p_1\cot z-4q_2+2p_2\left(\cot\frac{3}{2}z+\cot\frac{z}{2}\right)-6q_s+3p_3(\cot 2z+\cot z)\right\}\\&+\frac{i^2}{4}\left\{p_1\left(\frac{2}{3}n^2-2n+\frac{1}{3}+\frac{1}{\sin^2 z}\right)-q_1 n\cot z+4p_2\left(-2n+\frac{1}{\sin^2\frac{3}{2}z}+\frac{1}{\sin^2\frac{z}{2}}\right)\right.\\&\left.-4q_2\left(n\cot\frac{3}{2}z+n\cot\frac{z}{2}\right)+9p_3\left(-2n+\frac{1}{\sin^2 2z}+\frac{1}{\sin^2 z}\right)-9q_3(n\cot 2z+n\cot z)\right\}.\end{aligned}\right\}\quad(29)$$

$$\left.\begin{aligned}q_1=\mathrm{Q}_1&-\frac{i}{2}\left\{-q_1\cot z-p_1 n+2q_2\left(\cot\frac{z}{2}-\cot\frac{3}{2}z\right)+3q_3(\cot z-\cot 2z)\right\}\\&+\frac{i^2}{4}\left\{-p_1 n\cot z+q_1\left(\frac{2}{3}n^2+\frac{1}{3}-\frac{1}{\sin^2 z}\right)+4p_2\left(n\cot\frac{z}{2}-n\cot\frac{3}{2}z\right)\right.\\&\left.+4q_2\left(\frac{1}{\sin^2\frac{z}{2}}-\frac{1}{\sin^2\frac{3}{2}z}\right)+9p_3(n\cot z-n\cot 2z)+9q_3\left(\frac{1}{\sin^2 z}-\frac{1}{\sin^2 2z}\right)\right\}.\end{aligned}\right\}\quad(30)$$

$$\left.\begin{aligned}p_2=\mathrm{P}_2&-\frac{i}{2}\left\{-2q_1+p_1\left(\cot\frac{3}{2}z-\cot\frac{z}{2}\right)+2q_2(n-z)+2p_2(\cot 2z)-6q_3\right.\\&\left.+3p_3\left(\cot\frac{5}{2}z+\cot\frac{z}{2}\right)\right\}+\frac{i^2}{4}\left\{p_1\left(-2n+\frac{1}{\sin^2\frac{3}{2}z}+\frac{1}{\sin^2\frac{z}{2}}\right)-q_1\left(n\cot\frac{3}{2}z-n\cot\frac{z}{2}\right)\right.\\&+4p_2\left(\frac{2}{3}n^2-2n+\frac{1}{3}+\frac{1}{\sin^2 2z}\right)-4q_2 n\cot 2z+9p_3\left(-2n+\frac{1}{\sin^2\frac{5}{2}z}+\frac{1}{\sin^2\frac{z}{2}}\right)\\&\left.-9q_3\left(n\cot\frac{5}{2}z+n\cot\frac{z}{2}\right)\right\}.\end{aligned}\right\}\quad(31)$$

$$\left.\begin{aligned}q_2=\mathrm{Q}_2&-\frac{i}{2}\left\{q_1\left(-\cot\frac{z}{2}-\cot\frac{3}{2}z\right)-2q_2\cot 2z-2p_2 n+3q_3\left(\cot\frac{z}{2}-\cot\frac{5}{2}z\right)\right\}\\&+\frac{i^2}{4}\left\{p_1\left(-n\cot\frac{z}{2}-n\cot\frac{3}{2}z\right)+q_1\left(\frac{1}{\sin^2\frac{z}{2}}-\frac{1}{\sin^2\frac{3}{2}z}\right)-4p_2 n\cot 2z\right.\\&\left.+4q_2\left(\frac{2}{3}n^2+\frac{1}{3}-\frac{1}{\sin^2 2z}\right)+9p_3\left(n\cot\frac{z}{2}-n\cot\frac{5}{2}z\right)+9q_3\left(\frac{1}{\sin^2\frac{z}{2}}-\frac{1}{\sin^2\frac{5}{2}z}\right)\right\}.\end{aligned}\right\}\quad(32)$$

$$\left.\begin{aligned}
p_3=\mathrm{P}_3-\frac{i}{2}\Big\{&-2q_1+p_1(\cot 2z-\cot z)-4q_2+2p_2\left(\cot\frac{5}{2}z-\cot\frac{z}{2}\right)+3q_3(n-z)\\
&+3p_3\cot 3z\Big\}+\frac{i^2}{4}\Big\{p_1\left(-2n+\frac{1}{\sin^2 2z}+\frac{1}{\sin^2 z}\right)-q_1(n\cot 2z-n\cot z)\\
&+4p_2\left(-2n+\frac{1}{\sin^2\frac{5}{2}z}+\frac{1}{\sin^2\frac{z}{2}}\right)-4q_2\left(n\cot\frac{5}{2}z-n\cot\frac{z}{2}\right)\\
&+9p_3\left(\frac{2}{3}n^2-2n+\frac{1}{3}+\frac{1}{\sin^2 3z}\right)-9q_3 n\cot 3z\Big\}.
\end{aligned}\right\}\quad(33)$$

$$\left.\begin{aligned}
q_3=\mathrm{Q}_3-\frac{i}{2}\Big\{&q_1(-\cot z-\cot 2z)+2q_2\left(-\cot\frac{z}{2}-\cot\frac{5}{2}z\right)-3q_3\cot 3z-3p_3 n\Big\}\\
&+\frac{i^2}{4}\Big\{p_1(-n\cot z-n\cot 2z)+q_1\left(\frac{1}{\sin^2 z}-\frac{1}{\sin^2 2z}\right)+4p_2\left(-n\cot\frac{z}{2}-n\cot\frac{5}{2}z\right)\\
&+4q_2\left(\frac{1}{\sin^2\frac{z}{2}}-\frac{1}{\sin^2\frac{5}{2}z}\right)-9p_3 n\cot 3z+9q_3\left(\frac{2}{3}n^2+\frac{1}{3}-\frac{1}{\sin^2 3z}\right)\Big\}.
\end{aligned}\right\}\quad(34)$$

For the values of p_1, q_1, &c. in the last terms of equations (29) to (34), we must now insert their first approximations, $p_1=\mathrm{P}_1$, $q_1=\mathrm{Q}_1$, $p_2=\mathrm{P}_2$, $q_2=\mathrm{Q}_2$, &c. (and in the last terms but one, second approximations), as follows:—

$$\left.\begin{aligned}
p_1&=\mathrm{P}_1-\frac{i}{2}\Big\{\mathrm{Q}_1(n-2)+\mathrm{P}_1\cot z-4\,\mathrm{Q}_2+2\mathrm{P}_2\left(\cot\frac{3}{2}z+\cot\frac{z}{2}\right)\\
&\qquad\qquad-6\mathrm{Q}_3+3\mathrm{P}_3(\cot 2z+\cot z)\Big\};\\
q_1&=\mathrm{Q}_1-\frac{i}{2}\Big\{-\mathrm{Q}_1\cot z-\mathrm{P}_1 n+2\mathrm{Q}_2\left(\cot\frac{z}{2}-\cot\frac{3}{2}z\right)+3\mathrm{Q}_3(\cot z-\cot 2z)\Big\};\\
p_2&=\mathrm{P}_2-\frac{i}{2}\Big\{-2\mathrm{Q}_1+\mathrm{P}_1\left(\cot\frac{3}{2}z-\cot\frac{z}{2}\right)+2\mathrm{Q}_2(n-2)+2\mathrm{P}_2\cot 2z-6\mathrm{Q}_3\\
&\qquad\qquad+3\mathrm{P}_3\left(\cot\frac{5}{2}z-\cot\frac{z}{2}\right)\Big\};\\
q_2&=\mathrm{Q}_2-\frac{i}{2}\Big\{\mathrm{Q}_1\left(-\cot\frac{z}{2}-\cot\frac{3}{2}z\right)-2\mathrm{Q}_2\cot 2z-2\mathrm{P}_2 n+3\mathrm{Q}_3\left(\cot\frac{z}{2}-\cot\frac{5}{2}z\right)\Big\};\\
p_3&=\mathrm{P}_3-\frac{i}{2}\Big\{-2\mathrm{Q}_1+\mathrm{P}_1(\cot 2z-\cot z)-4\mathrm{Q}_2+2\mathrm{P}_2\left(\cot\frac{5}{2}z-\cot\frac{z}{2}\right)\\
&\qquad\qquad+3\mathrm{Q}_3(n-2)+3\mathrm{P}_3\cot 3z\Big\};\\
q_3&=\mathrm{Q}_3-\frac{i}{2}\Big\{\mathrm{Q}_1(-\cot z-\cot 2z)+2\mathrm{Q}_2\left(-\cot\frac{z}{2}-\cot\frac{5}{2}z\right)-3\mathrm{Q}_3\cot 3z-3\mathrm{P}_3\Big\}.
\end{aligned}\right\}\quad(35)$$

These operations correspond to the rejection of terms involving i^3.

We thus obtain, in lieu of equations (29) to (34), which involve the unknown true coefficients on both sides, others of the form

$$\left.\begin{aligned} p_1&=P_1+A_1i+\mathit{A}_1i^2,\\ q_1&=Q_1+B_1i+\mathit{B}_1i^2,\\ p_2&=P_2+A_2i+\mathit{A}_2i^2,\\ q_2&=Q_2+B_2i+\mathit{B}_2i^2,\\ p_3&=P_3+A_3i+\mathit{A}_3i^2,\\ q_3&=Q_3+B_3i+\mathit{B}_3i^2,\end{aligned}\right\} \quad \ldots \quad (36)$$
$$\&c.=\&c.,$$

in which A_1, A_1, B_1, B_1, &c. are numerical quantities.

The true period (and therefore i) being known, these expressions give the values of the coefficients for the true period in terms of those for the approximate period; and these values being inserted in equation (13), it will then express the phenomenon for the true period in terms of the coefficients for the approximate period. The general expression for A_1 and A_1 &c. would be too lengthy to write in full, although the calculation of their numerical values in any particular case is not very tedious; the most convenient mode of procedure is to work out, by equation (35), the *numerical* values of the second approximations to p_1, q_1, &c., and insert these in equations (29) to (34).

14. To illustrate the application of the method described, and to show that advantage is gained by it, we have chosen, arbitrarily, the law of periodical variation

$$\alpha_m=-\cos(mz+60°)+\cos 2mz-\cos 3mz,$$

or

$$\alpha_m=-\cdot5\cos mz+\cdot86603\sin mz+\cos 2mz-\cos 3mz,$$

where

$$p_1=-\cdot50000;\ q_1=+\cdot86603;\ p_2=+1\cdot00000;\ q_2=\cdot00000;\ p_3=+1\cdot00000;\ q_3=\cdot00000;$$

and taking $z=30$, $i=5'$, and $n=120$, we have calculated one hundred and twenty successive values of α_m, corresponding to the successive values of $z-0°\ 0'$, $30°\ 5'$, $60°\ 10'$, &c. . . . . $(3570°+9°\ 55')$; then, treating these numbers as if they corresponded to values of $z-0°$, $30°$, $60°$, &c. . . . . $3570°$, and applying to them Bessel's method, the following values of the approximate coefficients were obtained:—

$$P_1=-\cdot42262;\ Q_1=+\cdot90398;\ P_2=+\cdot97128,$$
$$Q_2=-\cdot17383;\ P_3=-\cdot96147;\ Q_3=+\cdot25536.$$

With these values, and the other data which supplied them, equations (35) and (36) give as third approximations to the true values of the coefficients,

$$p_1=-\cdot50000;\ q_1=+\cdot86604;\ p_2=+1\cdot00011;$$
$$q_2=+\cdot00002;\ p_3=-\cdot99995;\ q_3=+\cdot00002;$$

and as second approximations, that is excluding terms involving i^2,

$$p_1 = -\cdot 50089;\ q_2 = +\ \cdot 86829;\ p_3 = +1\cdot 01028;$$
$$q_2 = -\cdot 00328;\ p_3 = -1\cdot 02186;\ q_3 = +\cdot 00416;$$

the degree of approximation is in the second case close, and in the first almost perfect.

IV.

Application of the processes described to determine whether or not there be any periodical variation of disturbances of Magnetic Declination and Horizontal Force at Bombay, due to the influence of the planets Mercury, Venus, and the Earth, in the periods of their respective orbital revolutions, and of Mercury, Venus, and Jupiter in their synodic periods.*

15. In view of the remarkably definite evidence of periodicity in sun-spots indicative of planetary influence, brought to light by the investigations of Messrs. De La Rue, Stewart, and Loewy, and having regard to the common subjection of sun-spots and terrestrial magnetism to the well-known decennial period, it seemed to the writers very desirable to examine whether a similar connexion was exhibited by the two phenomena in respect of the planetary periods. The connexion was first shown to exist, by General Sir Edward Sabine, between the larger disturbances of terrestrial magnetism and sun-spots, but it has since been extended to include also the regular magnetic variations. The present inquiry will, however, be confined to the larger disturbances, and of these to the disturbances of Magnetic Declination and Horizontal Force at Bombay, of which a large body, extending over a period of twenty-six years, is available for use in the discussion.

16. A description of the Declinometer, and of the method adopted for separating disturbances, which is that of General Sabine, appears in the 'Philosophical Transactions,' 1869, pp. 363 to 368, and, like the Declinometer, the Horizontal-force Magnetometer is of the kind by Grubb of Dublin, originally supplied to the British Colonial Observatories. Disturbed observations of Declination (Easterly) may be defined as all those observations which give a value of the easterly declination *in excess* of the average of the remaining observations at the same hour during the same month by more than $1'\cdot 4$, and the easterly disturbance is that excess; and disturbed observations of Declination (Westerly) are all those observations which give a value of easterly declination *in defect* of that average by more than $1'\cdot 4$, and the westerly disturbance is that defect. In Table I. the aggregates of such excesses and defects are shown for each month in each of the twenty-six years from 1847 to 1872. Disturbed observations increasing the Horizontal Force are all those which give a value of the Horizontal Force *in excess* of the average of the remaining observations at the same hour during the same month by more

* The mathematical expression for the Earth's influence being analogous to the expressions for the influence of Mercury and Venus, the influences are here classed together indiscriminately, although doubtless they are not wholly of the same character in each case.

than ·00334 (metre-gramme-second) units of force, and the increasing disturbance is that excess; and the disturbed observations decreasing the Horizontal Force are all those which give a value of Horizontal Force *in defect* of that average by more than ·00334 units, and the decreasing disturbance is that defect. In Table II. the aggregates of the excesses and defects of Horizontal Force are shown for each month in each of the twenty-six years from 1847 to 1872. The two Tables contain all the observational data used in the present inquiry.

TABLE I.—Showing the Monthly Aggregates of Disturbances of Declination exceeding 1′·4 in amount, from January 1847 to December 1872.

Years.	Easterly Disturbance Aggregates.											
	Jan.	Feb.	March.	April.	May.	June.	July.	August.	Sept.	Oct.	Nov.	Dec.
1847.	32·601	9·313	53·343	81·730	52·675	31·843	41·527	69·892	153·729	108·458	67·142	210·564
1848.	52·444	82·755	84·714	46·396	25·086	87·569	87·888	44·773	15·669	81·189	233·638	27·256
1849.	66·268	94·313	55·685	30·832	16·206	16·850	18·278	32·054	19·469	53·973	92·298	20·561
1850.	28·997	28·544	34·003	15·067	18·623	16·777	42·915	21·585	15·093	8·540	5·443	82·517
1851.	74·813	21·741	13·837	20·075	20·990	22·818	35·098	21·894	90·523	49·580	18·995	72·240
1852.	53·346	129·985	24·127	74·522	126·439	14·392	18·704	34·921	32·894	16·508	28·006	36·853
1853.	32·650	24·427	42·940	25·270	67·160	39·312	27·949	36·882	94·948	19·424	17·374	38·480
1854.	50·902	64·843	56·504	43·466	10·708	22·208	31·258	10·283	22·554	44·416	18·769	22·813
1855.	13·100	7·185	36·425	19·414	8·449	14·416	13·078	10·702	16·942	14·624	14·734	14·338
1856.	8·054	14·978	10·926	13·924	1·665	12·098	5·052	9·554	21·379	15·433	0·000	6·893
1857.	8·118	6·752	20·732	8·924	10·489	13·311	33·118	12·057	25·447	20·650	40·744	96·264
1858.	49·958	39·181	64·664	111·087	18·894	25·834	77·967	30·488	43·782	44·923	27·337	15·034
1859.	57·819	69·377	48·189	97·001	52·618	38·394	57·820	114·162	169·905	198·777	15·435	49·717
1860.	23·984	70·715	103·117	37·567	17·695	38·733	140·932	221·409	64·623	63·580	11·436	38·406
1861.	60·495	56·850	37·678	33·454	19·002	41·033	39·174	32·787	47·383	40·953	35·443	69·321
1862.	30·153	26·618	179·782	84·432	27·592	21·266	51·978	81·257	55·343	163·152	13·866	55·289
1863.	62·003	39·771	60·408	56·068	56·448	19·299	27·583	32·972	27·227	22·576	12·177	0·000
1864.	17·328	32·728	26·264	27·672	6·541	80·336	57·361	40·140	39·891	20·698	5·020	20·390
1865.	52·565	27·247	9·906	39·858	27·211	25·372	53·231	70·072	30·130	41·462	50·878	13·940
1866.	30·095	81·696	16·635	3·045	17·569	13·226	3·059	27·330	12·088	31·926	32·221	8·047
1867.	30·424	15·881	4·994	2·875	14·832	11·655	16·841	25·108	21·224	40·817	27·571	0·000
1868.	6·695	4·370	16·389	110·398	5·117	18·591	13·281	38·992	55·874	85·009	7·087	6·428
1869.	55·697	43·308	60·588	75·824	67·852	52·912	17·095	30·644	116·304	50·840	17·692	59·819
1870.	64·717	57·199	70·171	101·645	77·381	64·724	33·539	62·790	79·164	153·136	51·690	45·709
1871.	8·155	10·946	9·078	15·883	3·705	5·598	14·513	11·408	6·347	6·087	12·137	2·614
1872.	3·818	18·097	8·803	6·006	7·109	8·368	10·140	14·941	10·221	31·311	3·622	6·856
	Westerly Disturbance Aggregates.											
1847.	22·397	7·678	0·000	44·942	27·333	39·353	28·176	40·743	60·499	31·620	37·967	53·274
1848.	58·531	54·526	11·946	27·527	22·894	33·295	54·331	53·568	6·745	46·017	81·873	17·325
1849.	47·932	29·114	51·404	50·760	21·203	39·349	40·501	49·756	32·347	28·293	26·837	11·849
1850.	11·865	62·015	13·427	33·615	39·415	11·933	41·324	24·019	23·114	10·068	4·772	16·425
1851.	25·920	26·131	16·060	6·148	36·951	48·112	40·230	41·939	26·847	23·471	7·637	10·712
1852.	31·093	43·683	19·581	42·771	21·890	11·452	39·386	24·997	35·918	24·430	28·512	17·105
1853.	71·474	2·889	23·612	13·683	35·860	27·882	26·281	23·721	24·766	42·571	6·748	8·598
1854.	49·328	6·672	26·553	9·388	12·748	34·806	16·006	8.158	8·245	18·580	12·791	4·506
1855.	8·686	14·803	1·447	13·382	6·207	30·243	21·586	23·924	23·347	5·025	3·191	5·548
1856.	16·100	9·312	0·000	7·282	9·634	6·364	27·199	6·584	25·866	3·252	8·740	3·050
1857.	36·052	12·035	5·744	7·974	36·935	12·845	22·012	10·628	97·241	28·370	13·757	57·011
1858.	48·242	14·885	36·986	12·387	18·998	24·157	31·895	35·153	61·432	38·204	32·387	21·273
1859.	29·810	31·247	33·086	35·032	54·457	34·145	41·192	49·006	81·670	43·294	22·539	45·387
1860.	45·737	24·530	59·494	53·723	46·127	39·309	171·465	59·967	31·190	35·831	16·268	23·089
1861.	23·738	12·540	28·022	44·486	19·973	36·941	64·533	31·395	20·087	45·476	10·469	10·644
1862.	24·502	11·317	39·491	99·643	20·131	35·380	52·467	36·508	32·164	96·376	18·179	19·360
1863.	45·074	29·515	24·148	35·883	17·075	9·793	28·931	29·248	10·118	29·466	12·512	9·864
1864.	16·426	21·826	13·412	17·374	17·771	19·065	25·817	37·766	20·984	9·253	8·052	6·418
1865.	12·802	19·926	11·832	3·695	48·677	8·625	25·480	44·368	38·070	34·283	3·015	3·355
1866.	36·845	38·828	17·246	20·491	205·299	2·929	17·898	41·593	6·695	70·411	3·457	2·902
1867.	14·021	6·881	5·481	3·156	3·080	1·811	13·432	23·159	21·280	11·930	3·073	0·000
1868.	1·523	14·104	25·252	34·643	12·828	20·381	16·917	19·469	56·602	149·506	1·482	0·000
1869.	20·950	6·277	13·912	32·983	21·287	21·938	39·260	20·195	55·086	19·379	8·246	33·442
1870.	23·818	19·064	32·187	41·901	54·201	63·784	90·833	72·874	40·028	30·685	42·759	28·250
1871.	4·967	6·338	6·090	4·773	3·473	5·347	6·644	8·570	4·551	5·588	3·172	2·443
1872.	2·720	4·309	5·640	6·245	4·697	3·534	2·892	6·209	4·559	2·592	4·422	3·794

TABLE II.—Showing the Monthly Aggregates of Disturbances of Horizontal Force exceeding ·00334 (metre-gramme-second) units of force, from January 1847 to December 1872.

Years.	January.	February.	March.	April.	May.	June.	July.	August.	September.	October.	November.	December.
	Disturbances increasing the Horizontal Force.											
1847.	·00907	·01775	·01674	·02732	·00000	·00000	·00000	·04421	·01062	·08554	·01976	·02142
1848.	·03658	·06152	·02606	·03002	·02851	·00000	·05591	·05134	·00000	·14069	·03931	·00000
1849.	·15325	·01714	·01512	·00709	·01523	·02326	·00745	·01570	·02459	·02833	·00659	·00418
1850.	·01577	·01544	·01105	·00000	·01620	·01951	·02412	·00000	·00727	·04946	·00364	·00796
1851.	·02077	·02365	·04298	·00000	·04550	·05728	·00349	·01577	·04406	·04360	·00000	·00749
1852.	·04536	·24419	·11981	·01501	·03380	·01676	·06502	·01202	·00403	·10634	·00832	·02668
1853.	·02796	·04004	·03449	·00369	·04307	·06008	·09990	·02997	·04000	·01945	·00431	·04752
1854.	·00409	·00756	·02712	·03405	·07026	·00000	·01132	·01723	·01548	·00708	·00000	·00701
1855.	·00000	·01460	·00748	·00000	·01445	·00000	·04876	·00000	·00000	·00339	·00361	·01106
1856.	·19274	·02631	·00000	·00389	·00349	·01745	·00000	·00763	·00348	·00393	·00000	·00768
1857.	·00000	·00000	·00788	·01888	·02353	·01179	·01651	·00000	·07689	·01406	·04776	·01950
1858.	·06034	·03265	·10996	·04717	·09666	·06326	·02741	·00000	·04577	·07583	·03001	·00974
1859.	·07124	·02244	·06345	·09157	·02721	·03326	·03232	·04560	·13183	·22485	·08819	·02266
1860.	·05302	·03254	·19500	·14087	·05049	·06765	·07483	·12618	·13884	·03510	·01262	·01841
1861.	·08372	·10390	·02560	·03140	·02096	·19531	·03583	·05782	·01107	·03264	·09290	·03801
1862.	·06612	·09045	·00368	·02728	·04650	·03443	·08801	·10445	·10420	·08087	·00769	·05626
1863.	·08151	·10808	·10098	·09064	·02881	·12260	·02140	·20006	·05847	·03759	·03886	·01101
1864.	·00372	·02653	·01799	·06810	·04201	·13497	·08710	·03553	·04273	·06378	·00788	·00000
1865.	·07666	·07456	·03871	·00721	·00338	·01185	·04881	·23497	·09785	·18215	·27939	·03966
1866.	·03058	·15329	·14081	·00829	·00746	·00000	·05271	·00382	·02139	·15274	·02206	·00685
1867.	·00492	·00768	·03120	·04817	·03156	·00752	·01156	·00730	·02848	·11074	·03429	·03356
1868.	·01109	·01866	·00729	·13932	·00364	·03920	·02604	·07877	·11978	·02688	·00000	·00000
1869.	·04751	·04613	·03694	·16159	·06674	·11497	·06630	·06377	·10358	·02430	·07031	·03387
1870.	·09332	·07691	·08896	·03309	·04273	·03987	·04036	·13553	·18851	·09500	·25815	·09626
1871.	·17428	·01968	·16671	·29371	·02320	·12342	·10727	·08968	·03199	·04688	·14534	·08743
1872.	·08526	·06741	·04145	·06270	·07961	·09175	·06115	·05030	·09116	·15568	·20619	·01625
	Disturbances decreasing the Horizontal Force.											
1847.	·13291	·07978	·33491	·47664	·41126	·01228	·03791	·24646	1·29535	·83941	·62093	1·32156
1848.	·41396	·73566	·52963	·44082	·25848	·00839	·42689	·04676	·14562	1·03536	1·31195	·14882
1849.	·15361	·34546	·07466	·04392	·06023	·11311	·02902	·01472	·09302	·15531	·42311	·04392
1850.	·05317	·12276	·06599	·02308	·04435	·07866	·11441	·02045	·04442	·22712	·01508	·08766
1851.	·50339	·21535	·06455	·00000	·18515	·14645	·06707	·08168	1·34014	·60210	·06286	·29423
1852.	·35381	1·05696	·11290	·41404	·12053	·19390	·11268	·02830	·28188	·11441	·14018	·08564
1853.	·04322	·20955	·15304	·28258	·44698	·14385	·28554	·00369	·68770	·02894	·16352	·38029
1854.	·20900	·15717	·33748	·42278	·11508	·03705	·08220	·01891	·09457	·26276	·06979	·04745
1855.	·01555	·05654	·08636	·10479	·08041	·02081	·07373	·02464	·01256	·25156	·00369	·02274
1856.	·01195	·05033	·05268	·03507	·04994	·00866	·01681	·09126	·06655	·10662	·00917	·12140
1857.	·00749	·02415	·00839	·10126	·47028	·04741	·02716	·00816	·24404	·05347	·35977	1·24577
1858.	·42793	·16029	·58995	·99566	·19593	·43089	·07337	·02714	·35608	·20237	·14029	·32211
1859.	·13537	·89391	·11752	·62330	·18915	·30410	·44571	·46413	1·12879	1·77052	·46265	·82666
1860.	·10656	·38462	1·09309	·38132	·31114	·08680	·94931	1·00879	·43830	·67307	·21485	·33536
1861.	·56006	·29277	·31021	·21158	·05457	·05590	·04264	·17087	·25154	·59525	·29297	·63135
1862.	·24050	·23634	·29742	·23392	·10651	·00891	·18965	·56125	·43924	·88875	·25039	·59343
1863.	·25914	·07879	·07900	·14163	·11356	·10037	·19459	·13433	·30873	·34188	·11432	·02252
1864.	·00339	·06082	·29916	·22083	·16067	·63418	·30811	·31811	·29252	·20128	·09667	·02541
1865.	·26233	·27368	·14320	·29201	·19018	·12277	·07451	·81050	·09827	·30819	·33401	·02432
1866.	·06729	·90626	·13067	·07426	·07078	·01029	·01222	·20720	·11476	·29021	·13876	·09057
1867.	·04851	·25234	·05092	·07295	·12774	·04279	·04308	·01123	·13665	·17425	·08676	·02294
1868.	·00000	·08876	·50612	·35348	·15057	·09218	·32487	·15639	·34444	·54636	·00000	·01414
1869.	·27997	·44794	·34134	·61314	·77931	·38806	·06795	·46084	·55401	·29192	·21115	·26690
1870.	·73891	·51550	·47810	·48246	·64513	·35981	·21999	·67844	1·28971	1·09525	·52101	·57811
1871.	·36799	1·12210	·51099	·72576	·05740	·27627	·33183	·44565	·31528	·37913	·73249	·04426
1872.	·05184	·96353	·15542	·92697	·21553	·27526	·51272	·94517	·53956	1·63515	·17688	·15274

17. The sidereal periods of revolution of Mercury, Venus, and the Earth are 87·97, 224·70, and 365·26 mean solar days respectively. Nine periods of Mercury are so nearly equal to $2\frac{1}{6}$ years (26 months) that the accumulated difference after ninety-nine periods is less than four days, or $\frac{1}{24}$ of one period of Mercury; and the time of thirteen periods of Venus differs from 8 years so little that after thirty-nine periods the accumu

lated difference is less than three days, or $\frac{1}{80}$ of the period of Venus; we shall therefore, in the first place, find (in accordance with what has preceded) the coefficients of BESSEL'S series expressing the variation of aggregate disturbance of Magnetic Declination, Easterly and Westerly, and the variation of aggregate disturbance of Horizontal Force (increasing and decreasing) with variation of the position of Mercury in its orbit, just as if the observations were wholly due to the action of that planet, and so for each planet in turn; and we shall afterwards examine to what extent the values of the coefficients thus found are affected by the influence of the other planets.

18. The ninety-nine periods of Mercury extend over 23 years and 10 months, and the observations treated commence with the aggregates of March 1847, and end with those of December 1870. The thirty-nine periods of Venus and twenty-four periods of the Earth extend over 24 years, the observations treated being those for January 1847 to December 1870.

The application of BESSEL'S process to these observations, taken from Tables I. and II., gives the values of the coefficients for the sidereal periods of Mercury, Venus, and the Earth as shown below*.

TABLE III.—Values of the coefficients p_1, q_1, &c. for the sidereal periods of Mercury, Venus, and the Earth.

Declination.						
	Easterly Disturbance.					
Coefficients	p_1.	q_1.	p_2.	q_2.	p_3.	q_3.
Mercury............	−0′·523	−1′·190	+3′·604	+3′·435	+3′·781	+2′·707
Venus...............	−4·591	−1·199	+1·428	+1·055	+3·422	−2·786
The Earth	+1·146	−4·054	−4·812	+7·217	+0·558	−0·700
	Westerly Disturbance.					
Coefficients	p_1.	q_1.	p_2.	q_2.	p_3.	q_3.
Mercury............	−4·302	−1·802	−1·314	+1·796	+4·533	−2·495
Venus...............	−1·516	+1·497	+1·156	−1·965	+2·440	+2·083
The Earth	−7·253	−1·168	+0·131	+2·584	+3·274	+0·684
Horizontal Force.						
	Increasing Disturbance.					
Coefficients	p_1.	q_1.	p_2.	q_2.	p_3.	q_3.
Mercury............	−·00702	+·00037	+·00080	+·00326	+·00322	−·00428
Venus...............	−·00493	+·00438	−·00203	+·00111	−·00190	+·00004
The Earth	−·00112	−·00524	−·00442	+·00967	+·00056	+·00779
	Decreasing Disturbance.					
Coefficients	p_1.	q_1.	p_2.	q_2.	p_3.	q_3.
Mercury............	−·00928	−·01847	+·02370	+·03871	+·05141	+·00490
Venus...............	−·02002	−·00659	−·00353	−·00730	+·01617	−·02442
The Earth	+·03189	−·06212	−·07347	+·04314	+·02711	+·01362

* An example of the calculations of one of these sets of coefficients is given at the end of the Appendix.

With these coefficients (and neglecting the non-periodic part of the phenomena) have been calculated the ordinates for the construction of the thick curves (Plate 53. figs. 1–12), the ordinates of which represent disturbance, and the abscissæ time.

19. It may be objected to the procedure thus far, that the application of BESSEL'S method to any arbitrary series of periodical numbers would yield a smooth-flowing curve, although the numbers themselves were subject to no corresponding law: this, we reply, is a mistake; the law is inherent in the series of numbers. It is another question to what cause the law must, in a particular case, be attributed; but this is so also when a periodical law has been found in a series of observations, by applying the common method of finding average values at different phases of the period. It may be interesting to some of our readers to show that, where the circumstances allow of the application of the latter method, it leads to the same form of curve as BESSEL'S process. We choose for this purpose the variations, with the sidereal period of Mercury, of disturbances of Declination (Easterly and Westerly) and of disturbances increasing and decreasing the Horizontal Force. If we take twenty-six equidistant times in the period of Mercury and twenty-six consecutive months, the several months will correspond to the twenty-six phases of Mercury's period, as shown below.

Twenty-sixths of the period of Mercury ...	0	1	2	3	4	5	6	7	8	9	10	11	12
Months ...	0	3	6	9	12	15	18	21	24	1	4	7	10
Twenty-sixths of the period of Mercury ...	13	14	15	16	17	18	19	20	21	22	23	24	25
Months ...	13	16	19	22	25	2	5	8	11	14	17	20	23

And arranging each successive twenty-six months' aggregates in this way and in successive lines, we get, for each phase, eleven observed disturbance-aggregates, of which averages are calculated. Means are then taken of each consecutive pair of these averages, forming twenty-six new averages, and this process is repeated six times; after this the means are taken of every consecutive three of the last averages, and these numbers are curved thin in figs. 1–4. It will be seen that they agree with the thick curves obtained by BESSEL'S process, which are also constructed from twenty-six equidistant ordinates; but the agreement is closer, as it clearly should be, when the twenty-six calculated ordinates are treated in the same manner (described above) as the twenty-six average disturbance-aggregates were, to obtain the ordinates of the thin curves. In this way the ordinates of the dotted curves have been obtained; and although the thick curves must be taken as best representing the true law, the dotted ones are more directly comparable with the thin curves, having been obtained by a similar process. The slight disagreement that is observable must be attributed mainly to the omission of the fourth and higher pairs of terms of BESSEL'S expression.

TABLE IV.—The observed and calculated values of Aggregate Disturbance of Declination are, for the sidereal period of Mercury, as follows:—

Easterly Disturbance Aggregates, diminished by the constant value 43′·180.

Twenty-sixths of the period	0	1	2	3	4	5	6	7	8	No.	Kind.
Observed	+3·588	+4·265	+4·259	+2·297	−1·197	−4·038	−4·880	−4·474	−3·926	2	Thin.
Calculated	+6·862	+8·622	+7·000	+2·746	−2·367	−6·326	−7·796	−6·628	−3·816	2	Thick.
Ditto rendered comparable with "observed"......	+4·401	+5·533	+4·566	+1·911	−1·383	−4·104	−5·392	−5·055	−3·557	2	Dotted.

Twenty-sixths of the period	9	10	11	12	13	14	15	16	17	No.	Kind.
Observed	−2·967	−0·968	+1·256	+2·217	+1·872	+1·455	+1·759	+2·841	+4·381	2	Thin.
Calculated	−0·855	+1·004	+1·364	+0·780	+0·346	+0·952	+2·748	+4·948	+6·233	2	Thick.
Ditto rendered comparable with "observed"......	−1·712	−0·242	+0·552	+0·866	+1·175	+1·874	+2·976	+4·038	+4·375	2	Dotted.

Twenty-sixths of the period	18	19	20	21	22	23	24	25	......	No.	Kind.
Observed	+5·054	+2·977	−1·792	−6·194	−6·892	−3·767	+0·236	+2·629	......	2	Thin.
Calculated	+5·482	+2·440	−2·012	−6·136	−8·127	−6·954	−2·924	+2·406	......	2	Thick.
Ditto rendered comparable with "observed"......	+3·454	+1·249	−1·632	−4·141	−5·235	−4·362	−1·759	+1·601	......	2	Dotted.

(No. and kind of corresponding figure.)

Westerly Disturbance Aggregates, diminished by the constant value 28′·557.

Twenty-sixths of the period	0	1	2	3	4	5	6	7	8	No.	Kind.
Observed	−4·218	−4·580	−4·231	−3·894	−3·739	−3·182	−1·705	+0·474	+2·394	1	Thin.
Calculated	−1·083	−3·195	−5·847	−7·421	−6·780	−3·889	+0·124	+3·516	+4·807	1	Thick.
Ditto rendered comparable with "observed"......	−2·531	−3·611	−4·867	−5·460	−4·804	−2·917	−0·442	+1·654	+2·567	1	Dotted.

Twenty-sixths of the period	9	10	11	12	13	14	15	16	17	No.	Kind.
Observed	+2·972	+1·923	+0·078	−1·252	−0·909	+1·793	+6·177	+ 9·662	+ 9·641	1	Thin.
Calculated	+3·554	+0·607	−2·277	−3·301	−1·545	+2·539	+7·311	+10·669	+11·070	1	Thick.
Ditto rendered comparable with "observed"......	+2·078	+0·688	−0·614	−0·838	+0·499	+3·103	+6·041	+ 8·140	+ 8·415	1	Dotted.

Twenty-sixths of the period	18	19	20	21	22	23	24	25	......	No.	Kind.
Observed	+6·123	+1·553	−1·755	−3·086	−2·880	−2·164	−2·108	−3·072	......	1	Thin.
Calculated	+8·239	+3·286	−1·822	−5·221	−5·980	−4·491	−2·171	−0·695	......	1	Thick.
Ditto rendered comparable with "observed"......	+6·573	+3·248	−0·302	−2·887	−3·955	−3·692	−2·828	−2·252	......	1	Dotted.

(No. and kind of corresponding figure.)

And for the sidereal periods of Venus and the Earth, the calculated values are:—

Easterly Disturbance Aggregates, diminished by the constant value 43′·026.

Twenty-fourths of the period	0	1	2	3	4	5	6	7
Venus	+0·259	−2·523	−5·752	−7·482	−6·555	−3·543	+0·159	+2·681
The Earth	−3·108	−0·602	+2·109	+4·271	+5·160	+4·256	+1·458	−2·764
Twenty-fourths of the period	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**
Venus	+3·049	+ 1·846	+ 0·408	+ 0·445	+2·597	+6·061	+9·010	+ 9·592
The Earth	−7·370	−10·993	−12·375	−10·822	−6·516	−0·516	+5·579	+10·163
Twenty-fourths of the period	**16**	**17**	**18**	**19**	**20**	**21**	**22**	**23**
Venus	+ 6·957	+ 2·113	−3·015	−6·219	−6·307	−3·956	−0·810	+0·985
The Earth	+12·152	+11·294	+8·166	+3·882	−0·318	−3·441	−4·937	−4·728

Westerly Disturbance Aggregates, diminished by the constant value 28′·463.

Twenty-fourths of the period	0	1	2	3	4	5	6	7
Venus	+2·080	+2·118	+0·373	−2·227	−4·220	−4·173	−1·742	+2·117
The Earth	−3·848	−3·105	−3·878	−5·201	−5·738	−4·627	−1·983	+1·177
Twenty-fourths of the period	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**
Venus	+5·640	+7·313	+6·483	+3·638	+0·232	−2·118	−2·653	− 1·703
The Earth	+3·586	+4·517	+4·208	+3·693	+4·110	+5·915	+8·484	+10·369
Twenty-fourths of the period	**16**	**17**	**18**	**19**	**20**	**21**	**22**	**23**
Venus	− 0·400	+0·133	−0·570	−2·117	−3·360	−3·383	−1·863	+0·402
The Earth	+10·084	+6·985	+1·721	−3·987	−8·192	−9·685	−8·554	−6·051

TABLE V.—The observed and calculated values of Aggregate Disturbance of Horizontal Force are, for the sidereal period of Mercury, as follows:—

Aggregates of Disturbances increasing the Horizontal Force, diminished by the constant value 0·4634.										No. and kind of corresponding figure.	
Twenty-sixths of the period......	0	1	2	3	4	5	6	7	8	No.	Kind.
Observed	−·00399	−·00525	−·00613	−·00500	−·00228	−·00003	+·00097	+·00168	+·00214	4	Thin.
Calculated	−·00300	−·00493	−·00677	−·00702	−·00507	−·00144	+·00239	+·00479	+·00492	4	Thick.
Ditto rendered comparable with "observed"......	−·00416	−·00481	−·00537	−·00508	−·00361	−·00128	+·00114	+·00278	+·00320	4	Dotted.
Twenty-sixths of the period......	9	10	11	12	13	14	15	16	17	No.	Kind.
Observed	+·00206	+·00177	+·00240	+·00410	+·00627	+·00840	+·00957	+·00890	+·00666	4	Thin.
Calculated	+·00307	+·00068	−·00049	+·00085	+·00460	+·00939	+·01303	+·01370	+·01061	4	Thick.
Ditto rendered comparable with "observed"......	+·00261	+·00178	+·00168	+·00292	+·00540	+·00826	+·01023	+·01024	+·00788	4	Dotted.
Twenty-sixths of the period......	18	19	20	21	22	23	24	25	......	No.	Kind.
Observed	+·00369	+·00021	−·00380	−·00725	−·00852	−·00735	−·00523	−·00387	......	4	Thin.
Calculated	+·00456	−·00239	−·00791	−·01044	−·00973	−·00696	−·00397	−·00247	......	4	Thick.
Ditto rendered comparable with "observed"......	+·00369	−·00114	−·00520	−·00748	−·00776	−·00664	−·00513	−·00417	......	4	Dotted.

Aggregates of Disturbances decreasing the Horizontal Force, diminished by the constant value ·27647.										No. and kind of corresponding figure.	
Twenty-sixths of the period......	0	1	2	3	4	5	6	7	8	No.	Kind.
Observed	+·03273	+·03976	+·03006	+·00077	−·03088	−·04195	−·03283	−·02474	−·02833	3	Thin.
Calculated	+·06583	+·06731	+·03956	−·00306	−·04144	−·06043	−·05604	−·03582	−·01417	3	Thick.
Ditto rendered comparable with "observed"......	+·03822	+·04076	+·02546	−·00006	−·02481	−·03974	−·04158	−·03354	−·02265	3	Dotted.
Twenty-sixths of the period......	9	10	11	12	13	14	15	16	17	No.	Kind.
Observed	−·02996	−·01902	−·00496	−·00039	+·00022	+·01175	+·03648	+·06147	+·07153	3	Thin.
Calculated	−·00344	−·00764	−·02008	−·02769	−·01843	+·01063	+·05108	+·08568	+·09700	3	Thick.
Ditto rendered comparable with "observed"......	−·01541	−·01401	−·01532	−·01297	−·00155	+·01955	+·04468	+·06395	+·06782	3	Dotted.
Twenty-sixths of the period......	18	19	20	21	22	23	24	25	......	No.	Kind.
Observed	+·05848	+·02410	−·01900	−·05104	−·05656	−·03673	−·00745	+·01676	......	3	Thin.
Calculated	+·07625	+·02852	−·02870	−·07265	−·08576	−·06350	−·01672	+·03363	......	3	Thick.
Ditto rendered comparable with "observed"......	+·05200	+·02030	−·01640	−·04451	−·05359	−·04103	−·01315	+·01760	......	3	Dotted.

And for the sidereal periods of Venus and the Earth, the calculated values are:—

Aggregates of Disturbances increasing the Horizontal Force, diminished by the constant value ·04611.

Twenty-fourths of the period	0	1	2	3	4	5	6	7
Venus	−·00886	−·00615	−·00209	+·00209	+·00520	+·00657	+·00637	+·00535
The Earth	−·00498	+·00447	+·01036	+·01029	+·00492	−·00260	−·00861	−·01088
Twenty-fourths of the period	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**
Venus	+·00440	+·00417	+·00453	+·00495	+·00480	+·00373	+·00199	+·00013
The Earth	−·00958	−·00667	−·00444	−·00383	−·00386	−·00247	+·00196	+·00905
Twenty-fourths of the period	**16**	**17**	**18**	**19**	**20**	**21**	**22**	**23**
Venus	−·00126	−·00195	−·00231	−·00293	−·00430	−·00639	−·00847	−·00957
The Earth	+·01624	+·01992	+·01745	+·00888	−·00274	−·01267	−·01672	−·01349

Aggregates of Disturbances decreasing the Horizontal Force, diminished by the constant value ·27529.

Twenty-fourths of the period	0	1	2	3	4	5	6	7
Venus	−·00738	−·03359	−·05313	−·05480	−·03645	−·00632	+·02136	+·03422
The Earth	−·01447	+·00146	+·01081	+·01223	+·00912	+·00465	−·00227	−·01667
Twenty-fourths of the period	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**
Venus	+·02855	+·01096	−·00581	−·01047	+·00032	+·02017	+·03697	+·04020
The Earth	−·04326	−·08081	−·11915	−·14164	−·13247	−·08558	−·00955	+·07405
Twenty-fourths of the period	**16**	**17**	**18**	**19**	**20**	**21**	**22**	**23**
Venus	+·02733	+·00514	−·01430	−·02080	−·01239	+·00364	+·01493	+·01165
The Earth	+·13906	+·16575	+·14921	+·10079	+·04200	−·00547	−·02903	−·02876

20. Eighty-two synodic periods of Mercury extend over 26 years, and the observations treated are those for the years 1847 to 1872. Fifteen synodic periods of Venus and twenty-two of Jupiter extend over 24 years, the years treated being 1847 to 1870.

TABLE VI.—Values of the coefficients p_1, q_1, &c. for the synodic periods of Mercury, Venus, and Jupiter.

Declination.

	Easterly Disturbance.					
Coefficients	p_1.	q_1.	p_2.	q_2.	p_3.	q_3.
Mercury	+0·719	−4·541	+0·776	−1·628	−3·122	+0·443
Venus	+0·356	−0·364	+3·521	−4·112	−7·920	−3·402
Jupiter	−3·334	−4·883	+1·361	−0·741	+1·916	−1·676
	Westerly Disturbance.					
Coefficients	p_1.	q_1.	p_2.	q_2.	p_3.	q_3.
Mercury	+1·197	−2·889	+1·775	+0·439	+2·553	−1·300
Venus	−1·604	−1·021	+1·160	+1·287	+0·278	−1·630
Jupiter	+0·251	+0·212	+0·189	−1·534	+0·806	+2·264

Table VI. (continued).

Horizontal Force.

	Increasing Disturbance.					
Coefficients	p_1.	q_1.	p_2.	q_2.	p_3.	q_3.
Mercury............	−·00553	−·00804	+·00127	−·00037	−·00335	+·00320
Venus...............	+·00672	−·00384	−·00075	−·00255	+·00537	−·00360
Jupiter	+·00150	−·00065	−·00556	+·00522	−·00146	−·00272

	Decreasing Disturbance.					
Coefficients	p_1.	q_1.	p_2.	q_2.	p_3.	q_3.
Mercury............	−·01019	−·02015	+·01047	+·00771	−·02165	−·00065
Venus...............	+·02607	−·02805	+·05284	−·03936	−·04866	−·02049
Jupiter	−·03509	−·03944	−·00751	+·02636	+·04341	+·01217

With these coefficients have been calculated the ordinates for the construction of the thick curves, Plate 54. figs. 13 to 24.

21. Table VII.—The calculated values of Aggregate Disturbance of Declination for the synodic periods of Mercury, Venus, and Jupiter are as follows:—

Easterly Disturbance Aggregates, diminished by the constant value shown in the last column of the Table.

Twenty-fourths of the period...	0	1	2	3	4	5	6	7	8
Mercury	−1·627	−2·517	−2·226	−1·810	−2·250	−3·793	−5·760	−6·951	−6·392
Venus................................	−4·043	−6·762	−5·077	−0·922	+2·462	+2·640	−0·483	−4·632	−6·612
Jupiter	−0·057	−3·507	−6·966	−9·090	−9·134	−7·300	−4·568	−2·122	− ·684
Twenty-fourths of the period...	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**
Mercury	−3·984	−0·652	+2·135	+ 3·179	+2·233	+0·182	−1·446	− 1·346	+ 0·821
Venus................................	−4·402	+1·429	+7·862	+11·085	+8·748	+1·475	−7·302	−13·104	−12·850
Jupiter	− ·184	+0·092	+ ·965	+ 2·779	+5·125	+7·042	+7·608	+ 6·490	+ 4·202
Twenty-fourths of the period...	**18**	**19**	**20**	**21**	**22**	**23**	**p_0.**		
Mercury	+4·208	+7·235	+ 8·436	+ 7·240	+4·248	+0·837	40′·472		
Venus................................	−6·559	+2·646	+10·214	+12·626	+9·213	+2·348	43′·026		
Jupiter	+1·846	+0·504	+ 0·608	+ 1·666	+2·552	+2·133	43′·026		

Westerly Disturbance Aggregates, diminished by the constant value shown in the last column of the Table.

Twenty-fourths of the period...	0	1	2	3	4	5	6	7	8
Mercury.............................	+5·525	+3·050	−0·440	−3·482	−4·964	−4·685	−3·364	−2·133	−1·814
Venus................................	−0·166	−1·120	−1·834	−1·918	−1·429	−0·808	−0·551	−0·870	−1·499
Jupiter	+1·246	+1·865	+1·353	−0·176	−1·919	−2·832	−2·241	−0·288	+2·099
Twenty-fourths of the period...	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**
Mercury.............................	−2·442	−3·274	−3·310	−1·975	+0·462	+2·974	+4·360	+3·950	+2·049
Venus................................	−1·831	−1·286	+0·298	+2·486	+4·416	+5·224	+4·492	+2·499	+0·084
Jupiter	+3·678	+3·575	+1·775	−0·868	−3·071	−3·821	−2·892	−0·925	+0·970
Twenty-fourths of the period...	**18**	**19**	**20**	**21**	**22**	**23**	**p_0.**		
Mercury.............................	−0·186	−1·379	−0·720	+1·564	+4·288	+5·946	26′·638		
Venus................................	−1·769	−2·426	−1·891	−0·744	+0·216	+0·426	28′·463		
Jupiter	+1·863	+1·494	+0·369	−0·610	−0·731	+0·087	28′·463		

Table VIII.—The calculated values of Aggregate Disturbance of Horizontal Force for the synodic periods of Mercury, Venus, and Jupiter are as follows:—

Aggregates of Disturbances increasing the Horizontal Force, diminished by the constant value shown in the last column of the Table.

Twenty-fourths of the period...	0	1	2	3	4	5	6	7	8
Mercury	−·00761	−·00661	−·00530	−·00533	−·00732	−·01037	−·01251	−·01189	−·00786
Venus	+·01134	+·00483	−·00228	−·00686	−·00718	−·00384	+·00051	+·00282	+·00126
Jupiter	−·00552	−·00387	·00000	+·00493	+·00895	+·01013	+·00763	+·00207	−·00451
Twenty-fourths of the period...	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**
Mercury	−·00151	+·00492	+·00917	+·01015	+·00845	+·00592	+·00459	+·00542	+·00781
Venus	−·00366	−·00950	−·01321	−·01284	−·00867	−·00288	+·00176	+·00350	+·00260
Jupiter	−·00969	−·01164	−·00993	−·00560	−·00053	+·00348	+·00551	+·00565	+·00471
Twenty-fourths of the period...	**18**	**19**	**20**	**21**	**22**	**23**	**p_0.**		
Mercury	+·00997	+·01005	+·00724	+·00225	−·00302	−·00661	·05000		
Venus	+·00099	+·00102	+·00390	+·00876	+·01318	+·01445	·04611		
Jupiter	+·00349	+·00233	+·00103	−·00075	−·00296	−·00491	·04611		

Aggregates of Disturbances decreasing the Horizontal Force, diminished by the constant value shown in the last column of the Table.

Twenty-fourths of the period...	0	1	2	3	4	5	6	7	8
Mercury	−·02137	−·01791	−·00763	+·00111	+·00056	−·01155	−·02997	−·04459	−·04592
Venus	+·03025	−·00489	−·01960	−·02085	−·02311	−·03690	−·06040	−·07984	−·07831
Jupiter	+·00081	−·00674	−·01886	−·04842	−·06853	−·06680	−·04410	−·01360	+·00771
Twenty-fourths of the period...	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**
Mercury	−·03053	−·00335	+·02469	+·04231	+·04375	+·03145	+·01431	+·00234	+·00111
Venus	−·04779	+·00342	+·05291	+·07543	+·05705	+·00426	−·05787	−·09791	−·09398
Jupiter	+·00986	−·00374	−·01808	−·01583	+·01150	+·05702	+·10114	+·12169	+·10616
Twenty-fourths of the period...	**18**	**19**	**20**	**21**	**22**	**23**	**p_0.**		
Mercury	+·00903	+·01875	+·02210	+·01511	+·00045	−·01425	·29213		
Venus	−·04528	+·02768	+·09365	+·12651	+·11760	+·07797	·27529		
Jupiter	+·05912	+·00024	−·04587	−·06258	−·04942	−·02128	·27529		

22. Let us now estimate the errors in the coefficients for the Earth due to the sidereal period of Venus, and those of the coefficients for the sidereal period of Venus due to the Earth's period. The periods have the ratio of 13 to 8, so that in equations (7) $f=13$, $g=8$, and $r=96$; that is, in 96 months eight periods of the Earth and thirteen of Venus have been just completed. The least value of s for which $(s\mp1)g$ or $(s\mp1)8$ is a multiple of r or 96 is 11*, and therefore p_{11} or q_{11} is the first coefficient (after p_1 or q_1) that affects the value of a_1 or b_1; the least value of s for which $(s\mp2)8$ is a multiple of 96 is 10, and therefore p_{10} or q_{10} is the first coefficient (after p_2 or q_2) that affects the value of a_2 or b_2; and similarly p_7 or q_7 is the first coefficient (after p_5 or q_5) that affects the value of a_5 or b_5. Hence if we may disregard as small those terms in the expression for the Earth's period which repeat themselves six or more times in a year, or whose period is less than two months, a_1, b_1, a_2, b_2, a_3, b_3, &c. will

* See second set of demonstrations in the Appendix.

each be affected by only one of the Earth's coefficients, viz. p_1, q_1, p_2, q_2, p_3, q_3, &c. respectively. Again, the least positive integral value of s for which $(sf\mp tg)$ or $(13s\mp 8t)$ is a multiple of r or 96 is 8; and therefore, if we may disregard as small those terms in the expression for the period of Venus which repeat themselves eight or more times in that period, the quantities a_1, b_1, a_2, b_2, a_3, b_3 &c., being unaffected by the disturbance due to the planet Venus, will sensibly represent the true coefficients of the expression for the Earth's disturbance variation. In a similar manner it may be shown that A_1, B_1, A_2, B_2, A_3, B_3 &c. of equations (7) are sensibly equal to the true coefficients of the expression for the period of Venus; for the least integral value of s for which $(s\mp t)f$ or $13\,(s\mp t)$ is a multiple of 96 is $s=96\mp t$, so that only very high terms, in the expression for Venus, would affect the values of the coefficients of the earlier terms; and further, since the least positive integral values of s and t which make $(sg\mp tf)$ or $8s\mp 13t$ a multiple of 96 are eleven and eight respectively, and the corresponding terms repeating themselves eleven and eight times respectively in the periods of the Earth and Venus, they may, as before, be neglected.

23. But we have adopted for thirteen periods of Venus the approximate time 8 years or 2922·05 days, instead of the true time 2921·11 days, which is less by 0·94 of a day. Having worked out the question in Section III. for three pairs of coefficients only, we will confine the examination to that number and to the Easterly disturbance variation for the sidereal period of the planet; and it will suffice that we determine *the second* approximations to the true coefficients, rejecting terms involving i^2, i. e. that we apply equation (35).

The first approximations are

$P_1=-4{\cdot}591$; $Q_1=-1{\cdot}199$; $P_2=+1{\cdot}428$; $Q_2=+1{\cdot}055$; $P_3=+3{\cdot}422$; $Q_3=-2{\cdot}786$; the angle

$$z=\frac{2c\pi}{n}=\frac{3\times 13}{3\times 96}\,2\pi=48^\circ\,45',$$

and the angle

$$i=\frac{\Delta\varkappa}{\varkappa}\,z=\frac{c\Delta\varkappa}{c\varkappa}\cdot\frac{2c\pi}{n}=\frac{c\Delta\varkappa}{\varkappa}\cdot\frac{2\pi}{n}=\frac{3\times 0{\cdot}94}{224{\cdot}7}\cdot\frac{2\pi}{3\times 96}=0'{\cdot}94\,;$$

and the greatest value of sni is $2\times(3\times 96)\times 0'{\cdot}94=9^\circ\,2'$ ($c\Delta\varkappa$ being the error in time in thirty-nine periods of Venus). Consequently sni being a small angle, the case is one to which the investigation in Section III. applies; therefore

$$\left.\begin{aligned}p_1&=P_1+A_1 i=-4{\cdot}591+{\cdot}044=-4{\cdot}547,\\ q_1&=Q_1+B_1 i=-1{\cdot}199-{\cdot}180=-1{\cdot}379,\\ p_2&=P_2+A_2 i=+1{\cdot}428-{\cdot}088=+1{\cdot}340,\\ q_2&=Q_2+B_2 i=+1{\cdot}055+{\cdot}115=+1{\cdot}170,\\ p_3&=P_3+A_3 i=+3{\cdot}422+{\cdot}330=+3{\cdot}752,\\ q_3&=Q_3+B_3 i=-2{\cdot}786+{\cdot}406=-2{\cdot}380,\end{aligned}\right\}$$

from which has been constructed the interrupted curve (Plate 53. fig. 6), which is seen at a glance to be almost identical with the thick curve constructed from the first approximations P_1, Q_1, &c.

24. We may now examine how the sidereal disturbance period of Mercury affects the

coefficients of that of the Earth or Venus, and *vice versâ*; for which purpose we must use equations (12) of Section II., viz.—

$$p_1=\frac{2}{R}\sum_{m=0}^{m=R-1}\left[\alpha_m \cos mz\right]-\frac{2}{R}\sum_{m=0}^{m=R-1}\left[P_s \cos s\frac{f}{g}mz+Q_s \sin s\frac{f}{g}mz\right]\cos mz,$$

which we will suppose to give the Earth's coefficient p_1, P_s and Q_s being the coefficients of Mercury, and $\frac{f}{g}$ being the ratio of the periods of the Earth and Mercury, which we may take as near enough to $\frac{108}{26}$ or $\frac{54}{13}$, $z=30°$, and $R=288$; inserting these values (12) becomes

$$p_1=\frac{1}{144}\sum_{m=0}^{m=287}\left[\alpha_m \cos mz\right]-\frac{1}{144}\sum_{m=0}^{m=287}\left[P_s \cos s\frac{54}{13}m\times 30°+Q_s \sin s\frac{54}{13}m\times 30°\right]\cos m\times 30°. \quad . \quad (37)$$

But the time of 314 observations is equal to 26 years, or 108 periods of Mercury, therefore

$$\sum_{m=0}^{m=313}\left[P_s \cos s\frac{54}{13}m\times 30°+Q_s \sin \frac{54}{13}m\times 30°\right]\cos m\times 30°=0; \quad . \quad . \quad . \quad . \quad (38)$$

and adding $\frac{1}{144}$ of this to (37), we have

$$p_1=\frac{1}{144}\sum_{m=0}^{m=287}\left[\alpha_m \cos m\,30°\right]+\frac{1}{144}\sum_{m=288}^{m=313}\left[P_s \cos s\frac{54}{13}m\times 30°+Q_s \sin s\frac{54}{13}m\times 30°\right]\cos m\times 30°; \quad (39)$$

and calculating the last term from the approximate coefficients of Mercury given in paragraph 18, we find its value to be, for Easterly disturbance, $+0{\cdot}006$; therefore

$$p_1=\frac{1}{144}\sum_{m=0}^{m=287}\left[\alpha_m \cos m\,30°\right]+0{\cdot}006.$$

Similarly we find

$$q_1=\frac{1}{144}\sum_{m=0}^{m=287}\left[\alpha_m \sin m\,30°\right]-0{\cdot}021,$$

$$p_2=\frac{1}{144}\sum_{m=0}^{m=287}\left[\alpha_m \cos 2m\,30°\right]+0{\cdot}071,$$

$$q_2=\frac{1}{144}\sum_{m=0}^{m=287}\left[\alpha_m \sin 2m\,30°\right]-0{\cdot}132;$$

and for Westerly disturbance

$$p_1=\frac{1}{144}\sum_{m=0}^{m=287}\left[\alpha_m \cos m\,30°\right]+0{\cdot}035,$$

$$q_1=\frac{1}{144}\sum_{m=0}^{m=287}\left[\alpha_m \sin m\,30°\right]+0{\cdot}061,$$

$$p_2=\frac{1}{144}\sum_{m=0}^{m=287}\left[\alpha_m \cos 2m\,30°\right]-0{\cdot}013,$$

$$q_2=\frac{1}{144}\sum_{m=0}^{m=287}\left[\alpha_m \sin 2m\,30°\right]-0{\cdot}101,$$

in all of which the last terms are small enough to be neglected, in comparison with the absolute range of any of the component periodical variations, as may be seen by simple inspection of the values of the several coefficients given in paragraph 18. And as these calculations are given more in illustration of the method than for any intrinsic value of the result, we need carry them no further.

25. Similarly, to find the effects of the Earth's period upon the coefficients for the sidereal period of Mercury, we have in lieu of (37),

$$p_1=\frac{1}{144}\sum_{m=0}^{m=287}\left[\alpha_m\cos\left(\frac{54}{13}m\,30^\circ\right)\right]-\frac{1}{144}\sum_{m=0}^{m=287}\left[\mathrm{P}_s\cos sm\,30^\circ+\mathrm{Q}_s\sin sm\,30^\circ\right]\cos\left(\frac{54}{13}m\,30^\circ\right)$$

$$=\frac{1}{144}\sum_{m=0}^{m=287}\left[\alpha_m\cos\left(\frac{54}{13}m\,30^\circ\right)\right]+\frac{1}{144}\sum_{m=288}^{m=313}\left[\mathrm{P}_s\cos sm\,30^\circ+\mathrm{Q}_s\sin sm\,30^\circ\right]\cos\left(\frac{54}{13}m\,30^\circ\right);$$

calculating which for Easterly disturbance, we obtain

$$p_1=\frac{1}{144}\sum_{m=0}^{m=287}\left[\alpha_m\cos\left(\frac{54}{13}m\,30^\circ\right)\right]-0\cdot016;$$

also

$$q_1=\frac{1}{144}\sum_{m=0}^{m=287}\left[\alpha_m\sin\left(\frac{54}{13}m\,30^\circ\right)\right]+0\cdot004,$$

$$p_2=\frac{1}{144}\sum_{m=0}^{m=287}\left[\alpha_m\cos 2\left(\frac{54}{13}m\,30^\circ\right)\right]-0\cdot022,$$

$$q_2=\frac{1}{144}\sum_{m=0}^{m=287}\left[\alpha_m\sin 2\left(\frac{54}{13}m\,30^\circ\right)\right]+0\cdot022,$$

$$p_3=\frac{1}{144}\sum_{m=0}^{m=287}\left[\alpha_m\cos 3\left(\frac{54}{13}m\,30^\circ\right)\right]-0\cdot031,$$

$$q_3=\frac{1}{144}\sum_{m=0}^{m=287}\left[\alpha_m\sin 3\left(\frac{54}{13}m\,30^\circ\right)\right]-0\cdot014;$$

and for Westerly disturbance

$$p_1=\frac{1}{144}\sum_{m=0}^{m=287}\left[\alpha_m\cos\left(\frac{54}{13}m\,30^\circ\right)\right]-0\cdot084,$$

$$q_1=\frac{1}{144}\sum_{m=0}^{m=287}\left[\alpha_m\sin\left(\frac{54}{13}m\,30^\circ\right)\right]+0\cdot000,$$

$$p_2=\frac{1}{144}\sum_{m=0}^{m=287}\left[\alpha_m\cos 2\left(\frac{54}{13}m\,30^\circ\right)\right]+0\cdot063,$$

$$q_2=\frac{1}{144}\sum_{m=0}^{m=287}\left[\alpha_m\sin 2\left(\frac{54}{13}m\,30^\circ\right)\right]-0\cdot032,$$

$$p_3=\frac{1}{144}\sum_{m=0}^{m=287}\left[\alpha_m\cos 3\left(\frac{54}{13}m\,30^\circ\right)\right]-0\cdot025,$$

$$q_3=\frac{1}{144}\sum_{m=0}^{m=287}\left[\alpha_m\sin 3\left(\frac{54}{13}m\,30^\circ\right)\right]+0\cdot046,$$

in all of which also the last terms are small enough to be neglected, in comparison with the range of each component variation, as may be seen from the values of the several coefficients given in paragraph 18.

26. To make a similar estimate of the reciprocal actions of Venus and Mercury would, with a month as the interval between successive observations, be extremely troublesome; but what has been done shows sufficiently the practicableness of the process, and we do not consider it necessary to apply it at present to this or any of the other cases we are dealing with.

27. The principal features pointed out by Messrs. De La Rue, Stewart, and Loewy* of the growth and decadence of sun-spots were of a simple character; the spots acquired a minimum magnitude at a heliocentric longitude a little greater than that of the planet, and a maximum at a heliocentric longitude a little more than 180° greater than that of the planet; and there was a gradual progression in the change from minimum to maximum and *vice versâ* in the intervening periods.

28. It must be admitted that the curves which we have found of magnetic variation in planetary periods do not possess the same simple character; but if we confine our attention to those of them which have been yielded by the largest number of individual observations of disturbance, viz. to the curves of *Easterly* disturbance of Declination and to the curves of disturbances decreasing the Horizontal Force, we shall find in them definitiveness of character and some remarkable points of correspondence and difference that would seem to be deserving of attention. We first note that, for the synodic period of Venus, the curves of Declination and Horizontal Force have their principal inflections alike, and that this likeness attaches, though in a less degree, to the curves for the synodic periods of Mercury and Jupiter, in common with those for Venus; secondly, that whilst the curves of Venus are strikingly bolder and more definite for the synodic period than for the sidereal period, there is no very marked difference in the case of the curves of Mercury. Again, we note the close resemblance in the two curves of the Earth and in the two of Mercury for its sidereal period—in the latter case of so precise a kind that, keeping in mind that the curves are derived from independent observations with instruments of different construction, it is difficult to suppose that they do not indicate a real periodicity in nature.

29. It is not claimed for these investigations that they account for any substantial part of the so-called decennial variation of magnetic disturbance, but only that there may be, and probably are, subordinate planetary variations of the kind described, which are superimposed upon the more strongly marked decennial variation, and that if they are, they are included with the variations that have been deduced from the observations.

It must be allowed, too, that, until the character of the decennial variation be brought out more fully than as yet (by a great extension of the period of observation), doubt must remain as to whether these apparent variations which follow the periods of the planets may not be due, wholly or in part, to the imperfect elimination of the decennial variation. The irregularities observed in the duration of the sun-spot period, with general correspondence in magnetic disturbance, as far as observation permits the comparison, would seem to indicate that the decennial period itself must be regarded as subordinate to some more extended period, in the recurrence of which the irregularities alluded to would be repeated in the same order. It is for this reason that we have not attempted, from the twenty-six years of observations available, to determine the duration and character of the decennial variation, considering that such an undertaking would, with present data, be to a great extent labour in vain.

* Proceedings of the Royal Society, vol. xx. page 210.

It is also because the decennial period would greatly affect the apparent variation of magnetic disturbance following the sidereal period of Jupiter, that no attempt has been made to apply these observations, extending over less than three such periods, to the determination of the character of that variation.

APPENDIX.

Demonstrations. First set.

To find the sum of each of the following series:—

(1) $\sin 0\beta + \sin\beta + \sin 2\beta + \ldots\ldots + \sin(n-1)\beta$.

(2) $\cos 0\beta + \cos\beta + \cos 2\beta + \ldots\ldots + \cos(n-1)\beta$.

(3) $0\sin 0\beta + \sin\beta + 2\sin 2\beta + \ldots\ldots + (n-1)\sin(n-1)\beta$.

(4) $0\cos 0\beta + \cos\beta + 2\cos 2\beta + \ldots\ldots + (n-1)\cos(n-1)\beta$.

(5) $0\sin 0\beta + \sin\beta + 2^2\sin 2\beta + \ldots\ldots + (n-1)^2\sin(n-1)\beta$.

(6) $0\cos 0\beta + \cos\beta + 2^2\cos 2\beta + \ldots\ldots + (n-1)^2\cos(n-1)\beta$.

If $X = 1 - 2x\cos\beta + x^2$, ... (a)

$$\frac{dX}{dx} = -2\cos\beta + 2x, \quad \text{(b)}$$

$$\frac{d^2X}{dx^2} = +2, \quad \text{(c)}$$

$$\frac{dX^2}{dx^2} = 4(x - \cos\beta)^2. \quad \text{(d)}$$

If $Y = x\sin(\alpha+\beta) - x^n\sin(\alpha+n\beta) + x^{n+1}\sin\{\alpha+(n-1)\beta\} - x^2\sin\alpha$, ... (e)

$$\frac{dY}{dx} = \sin(\alpha+\beta) - nx^{n-1}\sin(\alpha+n\beta) + (n+1)x^n\sin\{\alpha+(n-1)\beta\} - 2x\sin\alpha, \quad \text{(f)}$$

$$\frac{d^2Y}{dx^2} = -n(n-1)x^{n-2}\sin(\alpha+n\beta) + (n+1)nx^{n-1}\sin\{\alpha+(n-1)\beta\} - 2\sin\alpha. \quad \text{(g)}$$

And when $x=1$ and $\alpha=0$, these become respectively

$$X = 2(1-\cos\beta), \quad \text{(h)}$$

$$\frac{dX}{dx} = 2(1-\cos\beta), \quad \text{(i)}$$

$$\frac{d^2X}{dx^2} = +2, \quad \text{(j)}$$

$$\frac{dX^2}{dx^2} = 4(1-\cos\beta)^2, \quad \text{(k)}$$

$$Y = \sin\beta - \sin n\beta + \sin(n-1)\beta, \quad \text{(l)}$$

$$\frac{dY}{dx}=\sin\beta-n\sin n\beta+(n+1)\sin(n-1)\beta, \quad \text{(m)}$$

$$\frac{d^2Y}{dx^2}=-n(n-1)\sin n\beta+(n+1)n\sin(n-1)\beta. \quad \text{(n)}$$

When $x=1$ and $\alpha=\frac{\pi}{2}$, (a) to (d) have the same values as in (h) to (k) respectively, and (e) to (g) become as follows:—

$$Y=\cos\beta-\cos n\beta+\cos(n-1)\beta-1, \quad \text{(o)}$$

$$\frac{dY}{dx}=\cos\beta-n\cos n\beta+(n+1)\cos(n-1)\beta-2, \quad \text{(p)}$$

$$\frac{d^2Y}{dx^2}=-n(n-1)\cos n\beta+(n+1)n\cos(n-1)\beta-2. \quad \text{(q)}$$

Let $S=x\sin(\alpha+\beta)+x^2\sin(\alpha+2\beta)+x^3\sin(\alpha+3\beta)+\ldots x^{n-1}\sin\{\alpha+(n-1)\beta\}$. (1)

$$\sin(\alpha+n\beta+\beta)+\sin(\alpha+n\beta-\beta)=2\sin(\alpha+n\beta)\cos\beta,$$

$$2x^{n+1}\sin(\alpha+n\beta)\cos\beta=x^{n+1}\sin\{(\alpha+n\beta)+\beta\}+x^{n+1}\sin\{(\alpha+n\beta)-\beta\}.$$

Hence, by giving n the values 1, 2, 3 ..... $(n-1)$,

$$2x^2\sin\{\alpha+\beta\}\cos\beta=x^2\sin\{\alpha+2\beta\}+x^2\sin\alpha,$$

$$2x^3\sin\{\alpha+2\beta\}\cos\beta=x^3\sin\{\alpha+3\beta\}+x^3\sin\{\alpha+\beta\},$$

$$2x^4\sin\{\alpha+3\beta\}\cos\beta=x^4\sin\{\alpha+4\beta\}+x^4\sin\{\alpha+2\beta\},$$

&c. = &c. &c.,

$$2x^n\sin\{\alpha+(n-1)\beta\}\cos\beta=x^n\sin\{\alpha+n\beta\}+x^n\sin\{\alpha+(n-2)\beta\}.$$

Now adding

$$2xS\cos\beta=S-x\sin(\alpha+\beta)+x^n\sin\{\alpha+n\beta\}+x^2S-x^{n+1}\sin\{\alpha+(n-1)\beta\}+x^2\sin\alpha, \quad (2)$$

$$S(1-2x\cos\beta+x^2)=x\sin(\alpha+\beta)-x^n\sin(\alpha+n\beta)+x^{n+1}\sin\{\alpha+(n-1)\beta\}-x^2\sin\alpha, \quad (3)$$

which, when $x=1$ and $\alpha=0$, becomes

$$2S(1-\cos\beta)=\sin\beta-\sin n\beta+\sin(n-1)\beta, \quad (3a)$$

which, when $n\beta=2c\pi$,

$$\left.\begin{aligned}&=0, \text{ whether or not } \beta \text{ is 0 or a multiple of } 2\pi\\&=\sin\beta+\sin 2\beta+\sin 3\beta+\ldots+\sin(n-1)\beta.\end{aligned}\right\} \quad (3b)$$

If in (3) x be made $=1$ and $\alpha=\frac{\pi}{2}$,

$$2S(1-\cos\beta)=\cos\beta-\cos n\beta+\cos(n-1)\beta-1, \quad (3c)$$

which, when $n\beta=2c\pi$,

$$=-2(1-\cos\beta) \quad (3d)$$

$$S=-1=\cos\beta+\cos 2\beta+\cos 3\beta+\ldots+\cos(n-1)\beta,$$

to which adding $\cos 0\beta$, we have

$$\left.\begin{array}{ll} 0= & \cos 0+\cos\beta+\cos 2\beta+\cos 3\beta+\ldots\ldots+\cos(n-1)\beta, \\ \text{or}\quad n= & \text{do.} \end{array}\right\}$$

according as β is not or is 0 or a multiple of 2π;

$$\text{or (say) } SX=Y, \quad \ldots \quad (4)$$

$$S=YX^{-1}, \quad \ldots \quad (5)$$

$$\left.\begin{array}{l} \frac{dS}{dx}=\sin(\alpha+\beta)+2x\sin(\alpha+2\beta)+3x^2\sin(\alpha+3\beta)+\ldots\ldots \\ \qquad +(n-1)x^{n-2}\sin\{\alpha+(n-1)\beta\} \\ =\frac{dY}{dx}X^{-1}-Y\frac{dX}{dx}X^{-2}, \end{array}\right\} \quad \ldots \quad (6)$$

which, when $x=1$ and $\alpha=0$ (see equations (h), (i), (l), and (m),

$$\left.\begin{array}{l} =2^{-1}(1-\cos\beta)^{-1}\{\sin\beta-n\sin n\beta+(n+1)\sin(n-1)\beta\} \\ \qquad -2^{-2}(1-\cos\beta)^{-2}[2(1-\cos\beta)\{\sin\beta-\sin n\beta+\sin(n-1)\beta\}] \end{array}\right\} \quad . \quad . \quad (7)$$

$$=2^{-1}(1-\cos\beta)^{-1}[-(n-1)\sin n\beta+n\sin(n-1)\beta], \quad \ldots \quad (8)$$

which, when $n\beta=2c\pi$, c being an integer,

$$\left.\begin{array}{l} =-\frac{n\sin\beta}{4\sin^2\frac{\beta}{2}}=-\frac{n}{2}\cot\frac{\beta}{2} \\ =\sin\beta+2\sin 2\beta+3\sin 3\beta+\ldots\ldots+(n-1)\sin(n-1)\beta; \end{array}\right\} \quad \ldots \quad (9)$$

and as $0\sin 0\beta=0$,

$$\sum_{m=0}^{m=n-1} m\sin m\beta=-\frac{n}{2}\cot\frac{\beta}{2}, \text{ when } \beta \text{ is not 0 or a multiple of } 2\pi. \quad \ldots \quad (9a)$$

But when β is 0 or a multiple of 2π, each term of the series is 0, and

$$\sum_{m=0}^{m=n-1} m\sin m\beta=0. \quad \ldots \quad (9b)$$

Now let $x=1$ and $\alpha=\frac{\pi}{2}$, and (6) becomes

$$\left.\begin{array}{l} \frac{dS}{dx}=2^{-1}(1\cos\beta)^{-1}\{-\cos\beta-n\cos n\beta+(n+1)\cos(n-1)\beta\} \\ \qquad -2^{-2}(1-\cos\beta)^{-2}[2(1-\cos\beta)\{-\cos\beta-\cos n\beta+\cos(n-1)\beta+1\}] \end{array}\right\} \quad (10)$$

$$=2^{-1}(1-\cos\beta)^{-1}[-(n-1)\cos n\beta+n\cos(n-1)\beta-1],$$

which, when $n\beta=2c\pi$,

$$\left.\begin{array}{l} =2^{-1}(1-\cos\beta)^{-1}[-(n-1)+n\cos\beta-1]=-\frac{n}{2} \\ =\cos\beta+2\cos 2\beta+3\cos 3\beta+\ldots\ldots+(n-1)\cos(n-1)\beta; \end{array}\right\} \quad \ldots \quad (11)$$

and as $0 \cos 0\beta = 0$,

$$\sum_{m=0}^{m=n-1} m \cos m\beta = -\frac{n}{2}, \quad \text{(11}a\text{)}$$

except when β is 0 or a multiple of 2π, in which case

$$\sum_{m=0}^{m=n-1} m \cos m\beta = \{0+1+2+3+\ldots\ldots+(n-1)\} = \frac{n^2}{2} - \frac{n}{2}. \quad \text{(11}b\text{)}$$

Again, multiplying equation (6) by x,

$$\left.\begin{aligned} x\frac{dS}{dx} &= x\sin(\alpha+\beta)+2x^2\sin(\alpha+2\beta)+3x^3\sin(\alpha+3\beta)+\ldots\ldots \\ &\quad +(n-1)\,x^{n-1}\sin\{\alpha+(n+1)\beta\} \\ &= x\left\{\frac{dY}{dx}X^{-1}-Y\frac{dX}{dx}X^{-2}\right\}, \end{aligned}\right\} \quad (12)$$

$$\left.\begin{aligned} \frac{d}{dx}\left(x\frac{dS}{dx}\right) &= \sin(\alpha+\beta)+4x\sin(\alpha+2\beta)+9x^2\sin(\alpha+3\beta)+\ldots\ldots \\ &\quad +(n-1)^2x^{n-2}\sin\{\alpha+(n-1)\beta\} \\ &= \left\{\frac{dY}{dx}X^{-1}-Y\frac{dX}{dx}X^{-2}\right\}+x\left\{\frac{d^2Y}{dx^2}X^{-1}-\frac{dY}{dx}\cdot\frac{dX}{dx}X^{-2}-\frac{dY}{dx}\cdot\frac{dX}{dx}X^{-2}\right. \\ &\quad \left. -Y\frac{d^2X}{dx^2}X^{-2}+2Y\frac{dX^2}{dx^2}X^{-3}\right\} \end{aligned}\right\} \quad (13)$$

$$= X^{-1}\left\{\frac{dY}{dx}+x\frac{d^2Y}{dx^2}\right\}-X^{-2}\left\{\left(Y+2x\frac{dY}{dx}\right)\frac{dX}{dx}+xY\frac{d^2X}{dx^2}\right\}+2X^{-3}Yx\frac{dX^2}{dx^2}, \quad (14)$$

which, when $x=1$ and $\alpha=0$,

$$\begin{aligned} &= 2^{-1}(1-\cos\beta)^{-1}[\sin\beta-n^2\sin n\beta+(n+1)^2\sin(n-1)\beta] \\ &\quad -2^{-2}(1-\cos\beta)^{-2}[\{3\sin\beta-(2n+1)\sin n\beta+(2n+3)\sin(n-1)\beta\}2(1-\cos\beta) \\ &\qquad +2\{\sin\beta-\sin n\beta+\sin n-1)\beta\}] \\ &\quad +2^{-3}(1-\cos\beta)^{-3}[8(1-\cos\beta)^2\{\sin\beta-\sin n\beta+\sin(n-1)\beta\}], \end{aligned} \quad (15)$$

which, when $n\beta = 2c\pi$,

$$\begin{aligned} &= -2^{-1}(1-\cos\beta)^{-1}[\{n^2+2n\}\sin\beta] \\ &\quad -2^{-2}(1-\cos\beta)^{-2}[2(1-\cos\beta)\{-2n\sin\beta\}] \end{aligned} \quad (16)$$

$$= -2^{-1}(1-\cos\beta)^{-1}n^2\sin\beta = -\frac{2\sin\frac{\beta}{2}\cos\frac{\beta}{2}}{4\sin^2\frac{\beta}{2}}n^2 = -\frac{n^2}{2}\cot\frac{\beta}{2}$$

$$= \sin\beta+4\sin 2\beta+9\sin 3\beta+\ldots\ldots+(n-1)^2\sin(n-1)\beta; \quad (17)$$

and as $0^2 \sin 0\beta = 0$,

$$\sum_{m=0}^{m=n-1} m^2 \sin m\beta = -\frac{n^2}{2}\cot\frac{\beta}{2}, \text{ when } \beta \text{ is not 0 or a multiple of } 2\pi. \quad \text{(17}a\text{)}$$

But when β is 0 or a multiple of 2π, each term of the series is 0, and

$$\sum_{m=0}^{m=n-1} m^2 \sin m\beta = 0. \qquad (17b)$$

Now let $x=1$ and $\alpha=\frac{\pi}{2}$, and (14) becomes

$$\begin{aligned}\frac{d}{dx}\left(x\frac{dS}{dx}\right) &= 2^{-1}(1-\cos\beta)^{-1}[\cos\beta - n^2\cos n\beta + (n+1)^2\cos(n-1)\beta - 4] \\ &\quad - 2^{-2}(1-\cos\beta)^{-2}[\{3\cos\beta - (2n+1)\cos n\beta + (2n+3)\cos(n-1)\beta - 5\} \\ &\qquad \times 2(1-\cos\beta) + 2\{\cos\beta - \cos n\beta + \cos(n-1)\beta - 1\}] \\ &\quad + 2^{-3}(1-\cos\beta)^{-3}[8(1-\cos\beta)^2\{\cos\beta - \cos n\beta + \cos(n-1)\beta - 1\}], \qquad (18)\end{aligned}$$

which, when $n\beta = 2c\pi$,

$$\begin{aligned} &= 2^{-1}(1-\cos\beta)^{-1}[(n^2+2n+2)\cos\beta - n^2 - 4] \\ &\quad - 2^{-1}(1-\cos\beta)^{-1}[(2n+6)\cos\beta - (2n+6) - 2] \\ &\quad + 2^{-1}(1-\cos\beta)^{-1}[4\cos\beta - 4] \qquad (19) \\ &= 2^{-1}(1-\cos\beta)^{-1}[(n^2+2n+2-2n-6+4)\cos\beta - n^2 - 4 + 2n + 6 + 2 - 4], \qquad (20) \end{aligned}$$

$$\left.\begin{aligned} &= 2^{-1}(1-\cos\beta)^{-1}[-n^2(1-\cos\beta) + 2n] \\ &= -\frac{n^2}{2} + \frac{n}{2}\cdot\frac{1}{\sin^2\frac{\beta}{2}} \\ &= \cos\beta + 4\cos 2\beta + 9\cos 3\beta + \ldots + (n-1)^2\cos(n-1)\beta; \end{aligned}\right\} \qquad (21)$$

and as $0^2 \cos 0\beta = 0$,

$$\sum_{m=0}^{m=n-1} m^2 \cos m\beta = -\frac{n^2}{2} + \frac{n}{2}\frac{1}{\sin^2\frac{\beta}{2}}, \qquad (21a)$$

except when β is 0 or a multiple of 2π, in which case

$$\sum_{m=0}^{m=n-1} m^2 \cos m\beta = 0^2 + 1^2 + 2^2 + 3^2 + \ldots + (n-1)^2 = \frac{n^3}{3} - \frac{n}{2} + \frac{n}{6}. \qquad (21b)$$

Collecting together equations (3b), (3d), (9a), (9b), (11a), (11b), (17a), (17b), (21a), and (21b), we have, according as β is not or is 0 or a multiple of 2π,

$$\sum_{m=0}^{m=n-1} \sin m\beta = 0, \qquad \text{or } = 0,$$

$$\sum_{m=0}^{m=n-1} \cos m\beta = 0, \qquad \text{or } = n,$$

$$\sum_{m=0}^{m=n-1} m \sin m\beta = -\frac{n}{2}\cot\frac{\beta}{2}, \qquad \text{or } = 0,$$

$$\sum_{m=0}^{m=n-1} m \cos m\beta = -\frac{n}{2}, \qquad \text{or } = \frac{n^2}{2} - \frac{n}{2},$$

$$\sum_{m=0}^{m=n-1} m^2 \sin m\beta = -\frac{n^2}{2}\cot\frac{\beta}{2}, \qquad \text{or} = 0,$$

$$\sum_{m=0}^{m=n-1} m^2 \cos mb = -\frac{n^2}{2} + \frac{n}{2}\frac{1}{\sin^2\frac{\beta}{2}}, \qquad \text{or} = \frac{n^3}{3} - \frac{n^2}{2} + \frac{n}{6}.$$

Demonstrations. Second set.

To find when $(s \mp t)b$, $(sa \mp tb)$, $(s \mp t)a$, and $(sb \mp ta)$ are multiples of r if $a=13$, $b=8$, $r=96$, and s and t are positive and integral.

(1) $8(s \mp t) = 96c$, c being a positive integer, when

$$s \mp t = 12c,$$
$$s = 12c \pm t.$$

(2) $(13s \mp 8t) = 96c$, when

$$13s = 8(12c \pm t)$$
$$= 8\{13c - (c \mp t)\},$$
$$s = 8c - \tfrac{8}{13}(c \mp t),$$ which can only be integral when $(c \mp t)$ is a multiple of 13.

(3) $13(s \mp t) = 96c$

$$= 8(13-1)c,$$
$$s \mp t = 8(1 - \tfrac{1}{13})c,$$
$$s = 8\left(c - \frac{c}{13}\right) \pm t,$$ which can only be integral when c is a multiple of 13.

(4) $(8s \mp 13t) = 96c$, when

$$8s = 96c \pm 13t,$$
$$s = 12c \pm \tfrac{13}{8}t,$$ which can only be integral when t is a multiple of 8.

Specimen Calculation of BESSEL'S Coefficients, for variation of Aggregate

Successive months.	0	1	2	3	4	5	6	7	8	9	10	11	12
Twenty-sixths of the sidereal period of Mercury…………	0	9	18	1	10	19	2	11	20	3	12	21	4
Cosine …………	+1·000	−·567	−·355	+·971	−·749	−·120	+·885	−·885	+·120	+·749	−·971	+·355	+·568
Sine ……………	0·000	+·823	−·935	+·239	+·663	−·993	+·465	+·465	−·993	+·663	+·239	−·935	+·823

Monthly Aggregates

1st 9 period*……	53·343	81·730	52·675	31·843	41·527	69·892	153·729	108·458	67·142	210·564	52·444	82·755	84·714
2nd „ …	16·206	16·850	18·278	32·054	19·469	53·973	92·298	20·561	28·997	28·544	34·003	15·067	18·623
3rd „ …	35·098	21·894	90·523	49·580	18·995	72·240	53·346	129·985	24·127	74·522	126·439	14·392	18·704
4th „ …	94·948	19·424	17·374	38·480	50·902	64·843	56·504	43·466	10·708	22·208	31·258	10·283	22·554
5th „ …	14·734	14·338	8·054	14·978	10·926	13·924	1·665	12·098	5·052	9·554	21·379	15·433	0·000
6th „ …	49·958	39·181	64·664	111·087	18·894	25·834	77·967	30·488	43·782	44·923	27·337	15·034	57·819
7th „ …	103·117	37·567	17·695	38·733	140·932	221·409	64·623	63·580	11·436	38·406	60·495	56·850	37·678
8th „ …	27·592	21·266	51·978	81·257	55·343	163·152	13·866	55·289	62·003	39·771	60·408	56·068	56·448
9th „ …	57·361	40·140	39·891	20·698	5·020	20·390	52·565	27·247	9·906	39·858	27·211	25·372	53·231
10th „ …	12·088	31·926	32·221	8·047	30·424	15·881	4·994	2·875	14·832	11·655	16·841	25·108	21·224
11th „ …	7·087	6·428	55·697	43·308	60·588	75·824	67·852	52·912	17·095	30·644	116·304	50·840	17·692
Sums ……………	471·532	330·744	449·050	470·065	453·020	797·362	639·409	546·959	295·080	550·649	574·119	367·202	388·687
Means……………	42·867	30·068	40·823	42·733	41·184	72·487	58·128	49·724	26·825	50·059	52·193	33·382	35·335
Variations ………	−0·313	−13·112	−2·357	−0·447	−1·996	+29·307	+14·948	+6·544	−16·355	+6·879	+9·013	−9·798	−7·845

* Commencing with March 1847.

Easterly Disturbance of Declination in the Sidereal Period of Mercury.

13	14	15	16	17	18	19	20	21	22	23	24	25	Successive Months.
13	22	5	14	23	6	15	24	7	16	25	8	17	Twenty-sixths of the sidereal period of Mercury.
−1·000	+·568	+·355	−·971	+·749	+·120	−·885	+·885	−·120	−·749	+·971	−·355	−·568	Cosine.
0·000	−·823	+·935	−·239	−·663	+·993	−·465	−·465	+·993	−·663	−·239	+·935	−·823	Sine.

of Disturbance.

46·396	25·086	87·569	87·888	44·773	15·669	81·189	233·638	27·256	66·268	94·313	55·685	30·832	1st 9 period*.
16·777	42·915	21·585	15·093	8·540	5·443	82·517	74·813	21·741	13·837	20·075	20·990	22·818	2nd „
34·921	32·894	16·508	28·006	36·853	32·650	24·427	42·940	25·270	67·160	39·312	27·949	36·882	3rd „
44·416	18·769	22·813	13·100	7·185	36·425	19·414	8·449	14·416	13·078	10·702	16·942	14·624	4th „
6·893	8·118	6·752	20·732	8·924	10·489	13·311	33·118	12·057	25·447	20·650	40·744	96·264	5th „
69·377	48·189	97·001	52·618	38·394	57·820	114·162	169·905	198·777	15·435	49·717	23·984	70·715	6th „
33·454	19·002	41·033	39·174	32·787	47·383	40·953	35·443	69·321	30·153	26·618	179·782	84·432	7th „
19·299	27·583	32·972	27·227	22·576	12·177	0·000	17·328	32·728	26·264	27·672	6·541	80·336	8th „
70·072	30·130	41·462	50·878	13·940	30·095	81·696	16·635	3·045	17·569	13·226	3·059	27·330	9th „
40·817	27·571	0·000	6·695	4·370	16·389	110·398	5·117	18·591	13·281	38·992	55·874	85·009	10th „
59·819	64·717	57·199	70·171	101·645	77·381	64·724	33·539	62·790	79·164	153·136	51·690	45·709	11th „
442·241	344·974	424·894	411·582	319·987	341·921	632·791	670·925	485·992	367·656	494·413	483·240	594·951	Sums.
40·204	31·361	38·627	37·417	29·090	31·084	57·526	60·993	44·181	33·423	44·947	43·931	54·086	Means.
−2·976	−11·819	−4·553	−5·763	−14·090	−12·096	+14·346	+17·813	+1·001	−9·757	+1·767	+0·751	+10·906	Variations.

* Commencing with March 1847.

Specimen Calculation (continued).

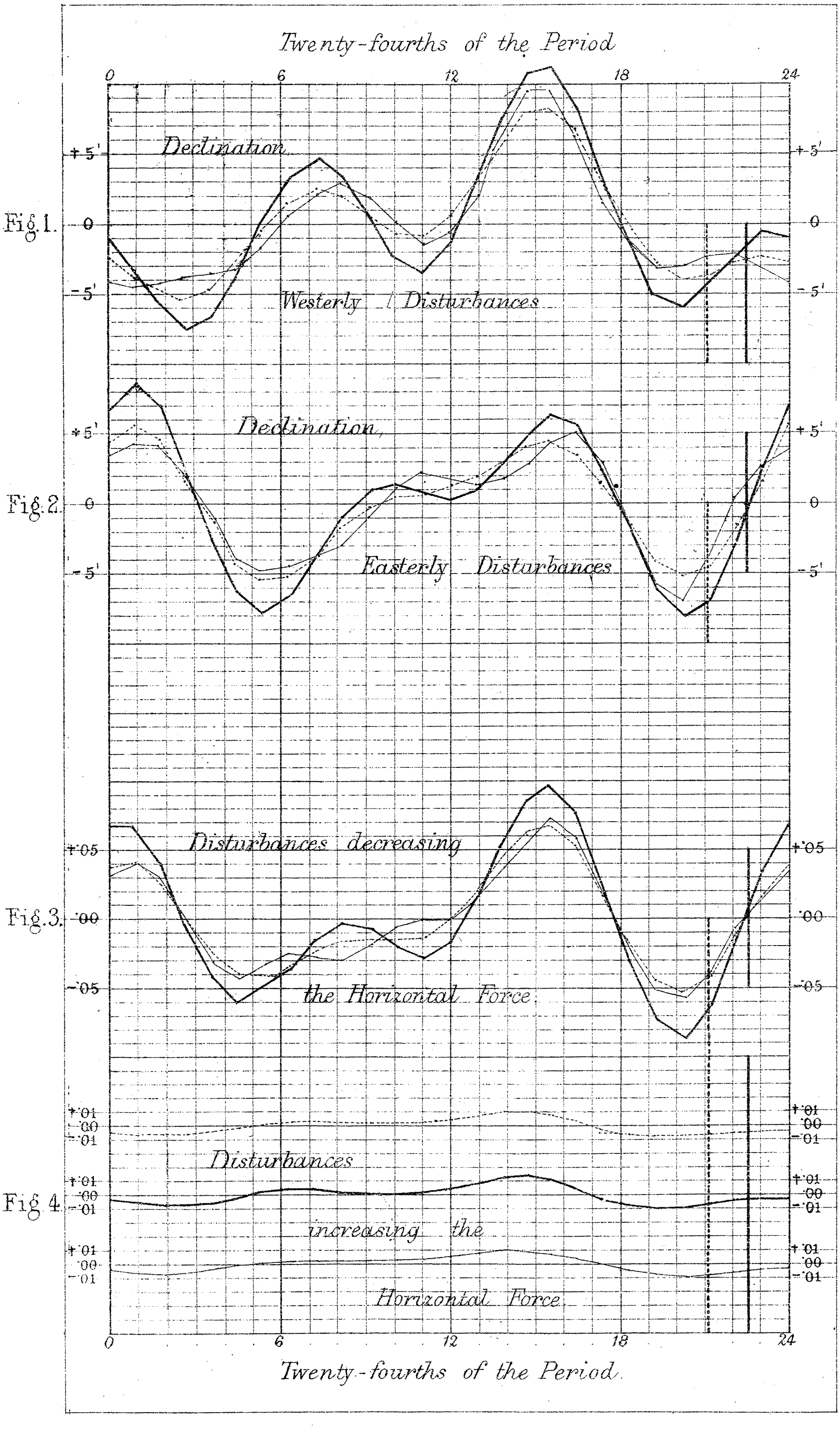
Symbol of operation.							
				P_1.			
+(0 to 6) ……	− 0·313	−13·112	− 2·357	−0·447	− 1·996	+29·307	+14·948
−(13 to 7)……	+ 2·976	+ 7·845	+ 9·798	−9·013	− 6·879	+16·355	− 6·544
−(14 to 19) …	……	+11·819	+ 4·553	+5·763	+14·090	+12·096	−14·346
+(25 to 20) …	……	+10·906	+ 0·751	+1·767	− 9·757	+ 1·001	+17·813
Sums ………	+ 2·663	+17·458	+12·745	−1·930	− 4·542	+58·759	+11·871
Factors ………	+ 1·000	− 0·568	− 0·355	+0·971	− 0·749	− 0·120	+ 0·885
Products ……	+ 2·663	− 9·916	− 4·524	−1·874	+ 3·402	− 7·051	+10·506
Sum of Products ……	− 6·794						
Ditto ÷ 13 ……	− 0·523	$=P_1$.					
				P_2.			
+(0 to 6) ……	− 0·313	−13·112	− 2·357	−0·447	− 1·996	+29·307	+14·948
+(13 to 7)……	− 2·976	− 7·845	− 9·798	+9·013	+ 6·879	−16·355	+ 6·544
+(14 to 19) …	……	−11·819	− 4·553	−5·763	−14·090	−12·096	+14·346
+(25 to 20) ..	……	+10·906	+ 0·751	+1·767	− 9·757	+ 1·001	+17·813
Sums ………	− 3·289	−21·870	−15·957	+4·570	−18·964	+ 1·857	+53·651
Factors ………	+ 1·000	− 0·355	− 0·749	+0·885	+ 0·121	− 0·971	+ 0·568
Products ……	− 3·289	+ 7·764	+11·952	+4·044	− 2·295	− 1·803	+30·474
Sum of Products ……	+46·847						
Ditto ÷ 13 ……	+ 3·604	$=P_2$.					
				P_3.			
Sum* ………	+ 2·663	+17·458	+12·745	−1·930	− 4·542	+58·759	+11·871
Factors ………	+ 1·000	+ 0·971	+ 0·885	+0·749	+ 0·568	+ 0·355	+ 0·120
Products ……	+ 2·663	+16·952	+11·279	−1·446	− 2·580	+20·859	+ 1·425
Sum of Products ……	+49·152						
Ditto ÷ 13 ……	+ 3·781	$=P_3$.					

				Q_1.			
+(0 to 6) ……	− 0·313	−13·112	−2·357	− 0·447	− 1·996	+29·307	+14·948
+(13 to 7)……	− 2·976	− 7·845	−9·798	+ 9·013	+ 6·879	−16·355	+ 6·544
−(14 to 19) …	……	+11·819	+4·553	+ 5·763	+14·090	+12·096	−14·346
−(25 to 20) …	……	−10·906	−0·751	− 1·767	+ 9·757	− 1·001	−17·813
Sums ………	− 3·289	−20·044	−8·353	+12·562	+28·730	+24·047	−10·667
Factors ………	0·000	+ 0·823	−0·935	+ 0·239	+ 0·663	− 0·993	+ 0·465
Products ……	0·000	−16·496	+7·810	+ 3·002	+19·048	−23·879	− 4·960
Sum of Products ……	−15·475						
Ditto ÷ 13 ……	− 1·190	$=Q_1$.					
				Q_2.			
+(0 to 6) ……	− 0·313	−13·112	−2·357	− 0·447	− 1·996	+29·307	+14·948
−(13 to 7)……	+ 2·976	+ 7·845	+9·798	− 9·013	− 6·879	+16·355	− 6·544
+(14 to 19) …	……	−11·819	−4·553	− 5·763	−14·090	−12·096	+14·346
−(25 to 20) …	……	−10·906	−0·751	− 1·767	+ 9·757	− 1·001	−17·813
Sums ………	+ 2·663	−27·992	+2·137	−16·990	−13·208	+32·565	+ 4·937
Factors ………	0·000	− 0·935	+0·663	+ 0·465	− 0·993	+ 0·239	+ 0·823
Products ……	0·000	+26·173	+1·417	− 7·900	+13·116	+ 7·783	+ 4·063
Sum of Products ……	+44·652						
Ditto ÷ 13 ……	+ 3·435	$=Q_2$.					
				Q_3.			
Sum* ………	− 3·289	−20·044	−8·353	+12·562	+28·730	+24·047	−10·667
Factors ………	0·000	+ 0·239	+0·465	+ 0·663	+ 0·823	+ 0·935	+ 0·993
Products ……	0·000	− 4·791	−3·884	+ 8·329	+23·645	+22·484	−10·592
Sum of Products ……	+35·191						
Ditto ÷ 13 ……	+ 2·707	$=Q_3$.					

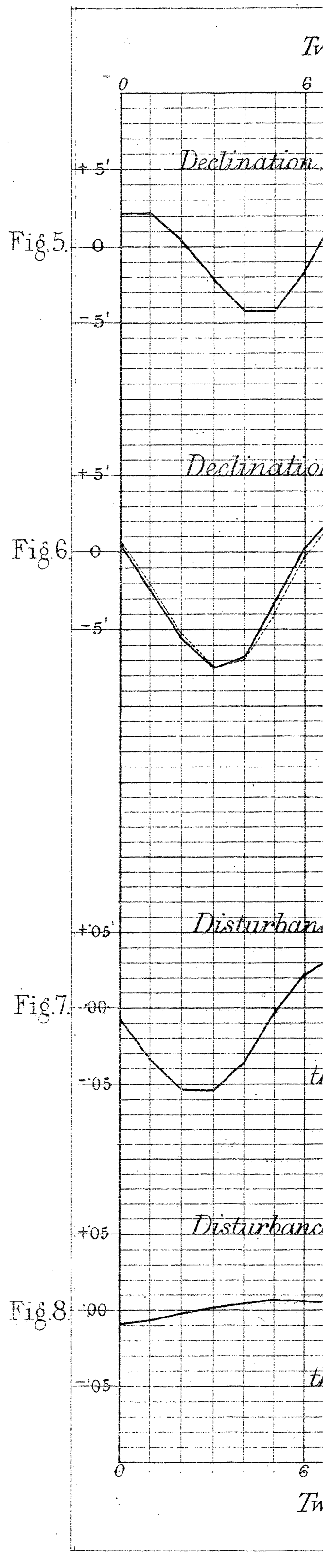
* These sums are the same as those in the 16th line above them.

Disturbance Variations of Magnetic Declination and H

MERCURY.



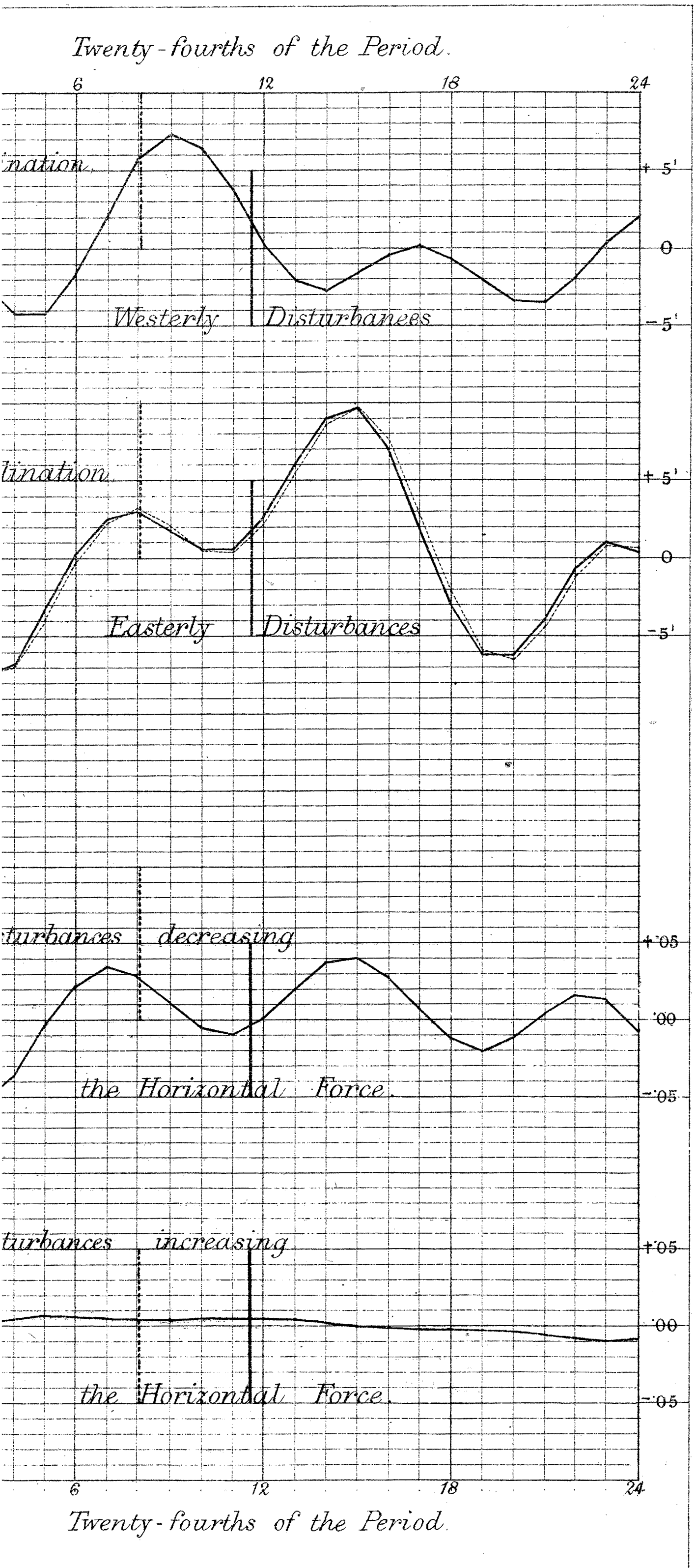
The vertical dotted and thick lines mark the times of the ascending Node and Perihelion respectively.



The vertical dotte… ascending Node…

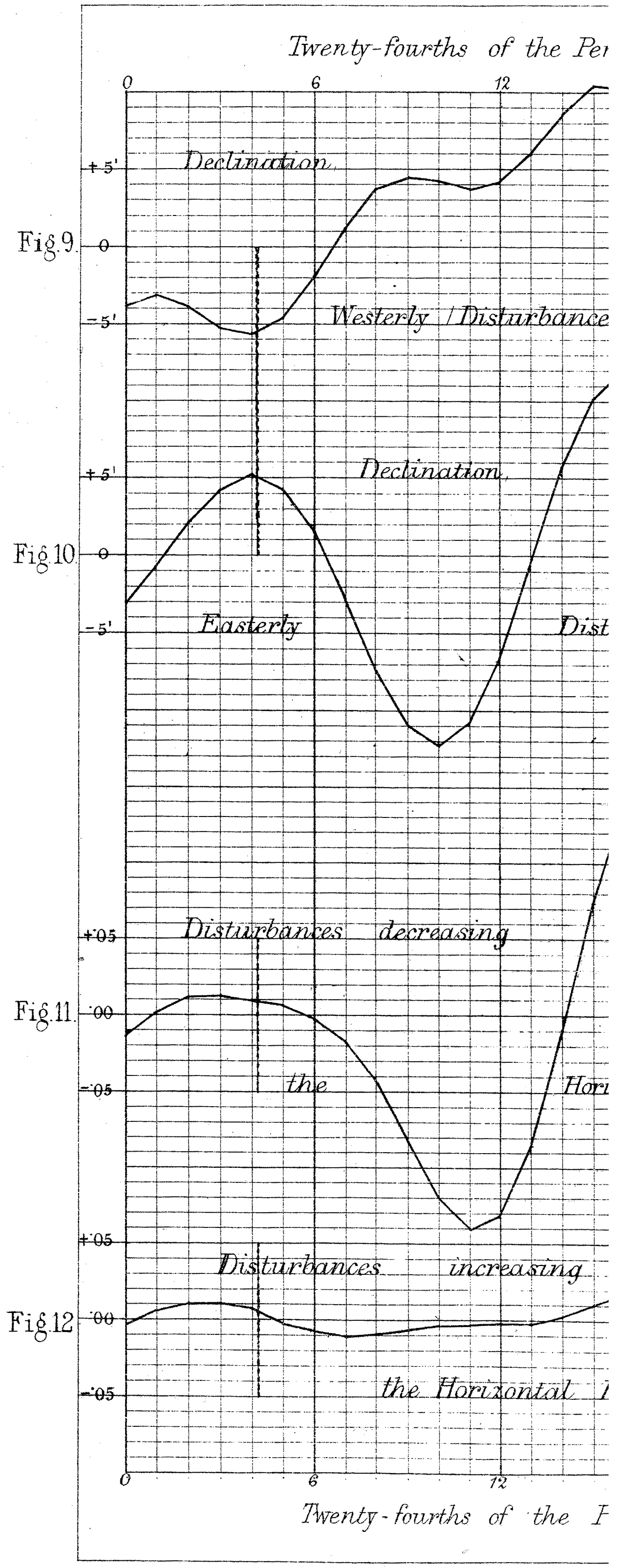
nd Horizontal Force in the Sidereal Periods of Mercury, Venus and the

VENUS.



l dotted and thick lines mark the times of the
Node and Perihelion respectively.

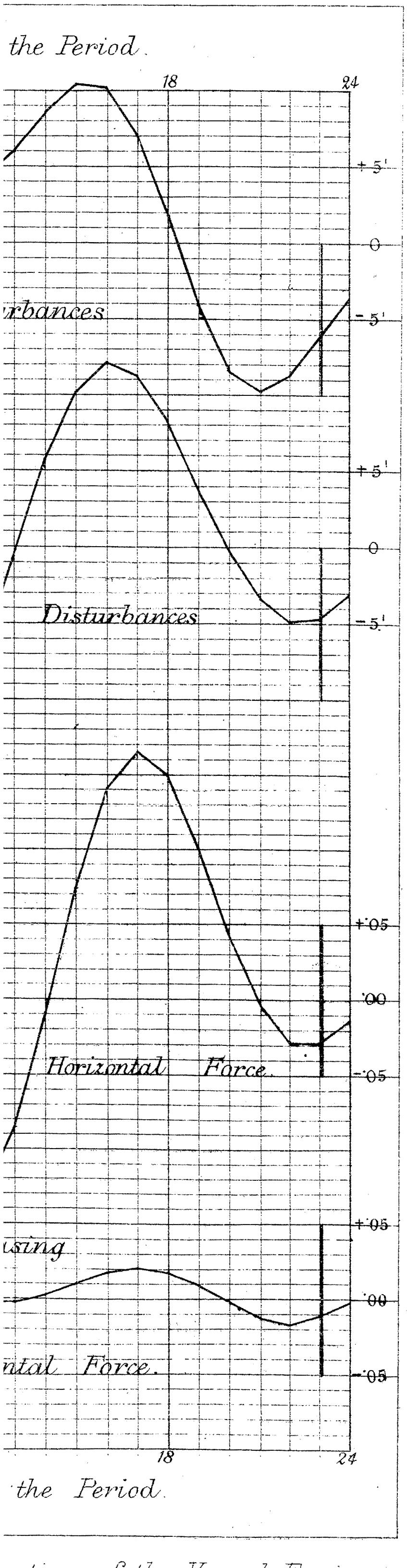
THE EART



The Vertical dotted lines mark the time
and the Vertical thick lines the time

the Earth.

ARTH.



e time of the Vernal Equinox,
time of Perihelion.

The vertical dotted and thick lines mark the times of the ascending Node and Perihelion respectively.

The vertical dotte
ascending Node

W. H. Wesley lith.

l dotted and thick lines mark the times of the Node and Perihelion respectively.

The Vertical dotted lines mark the time and the Vertical thick lines the time

e time of the Vernal Equinox,
time of Perihelion.

W. West & Co. imp.

Disturbance Variations of Magnetic Declination and ...

MERCURY.

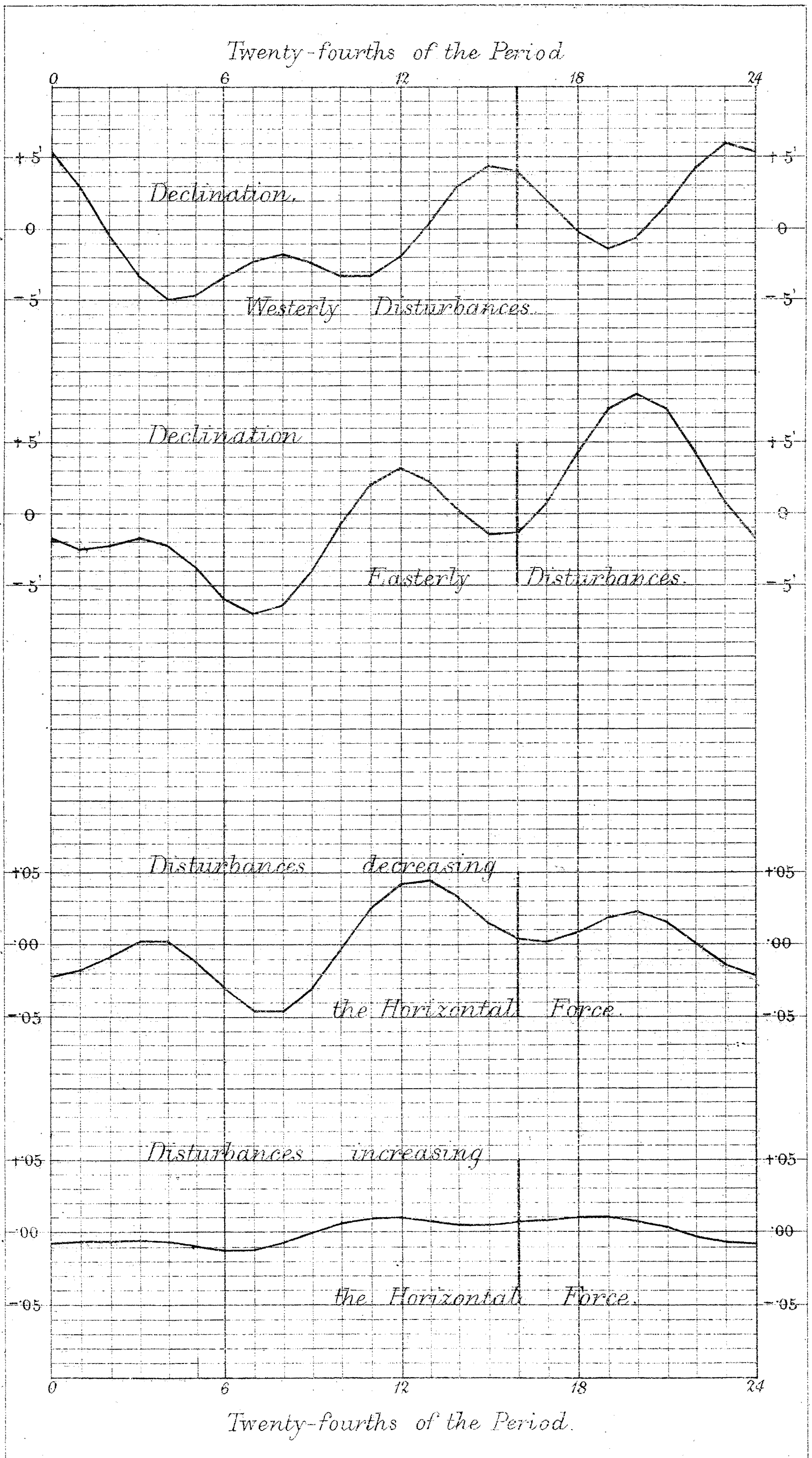


Fig. 13. Fig. 14. Fig. 15. Fig. 16.

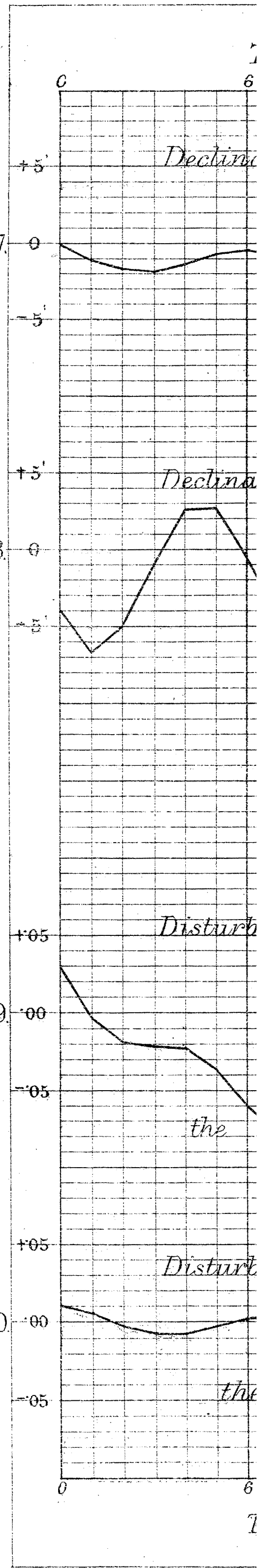


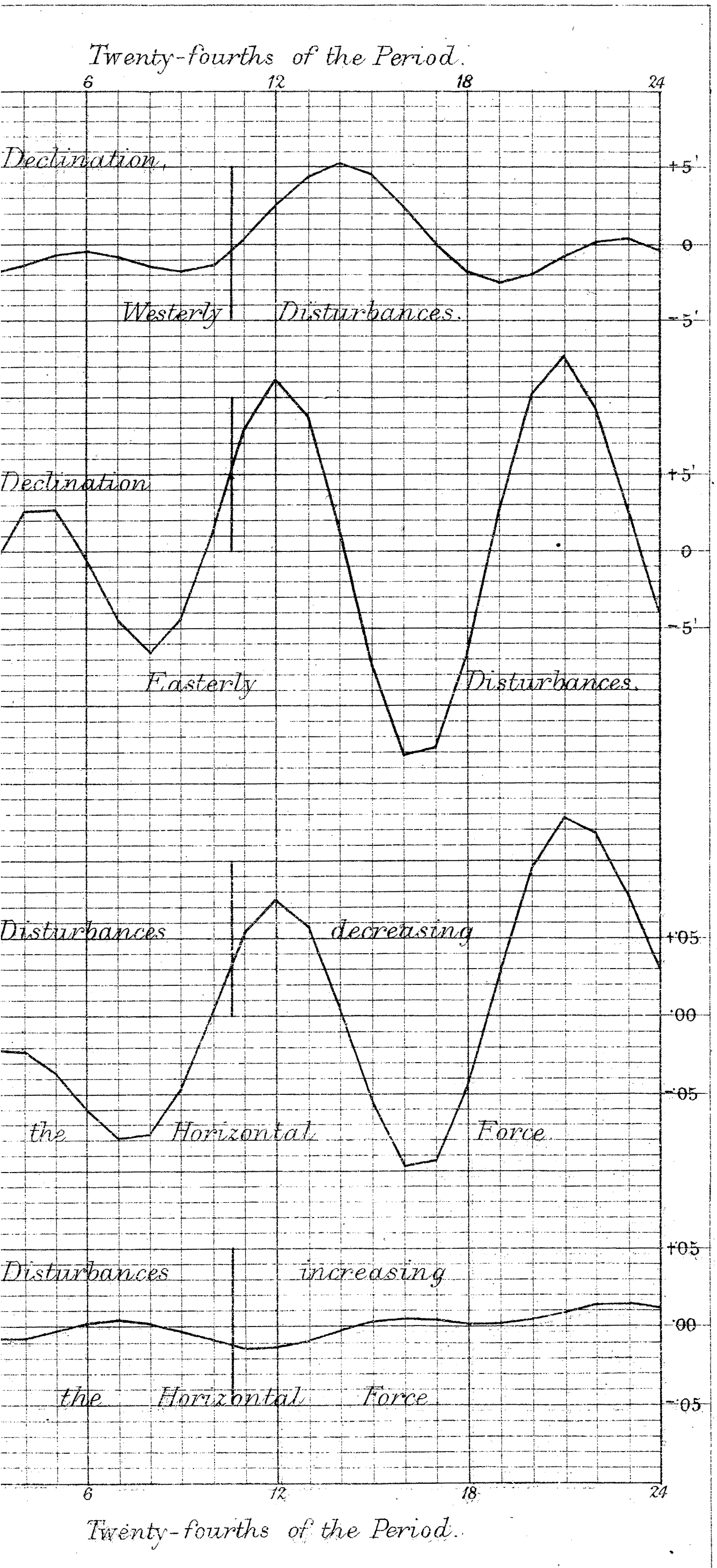
Fig. 17. Fig. 18. Fig. 19. Fig. 20.

The vertical thick lines mark the time of inferior Conjuntion.

The vertical thi

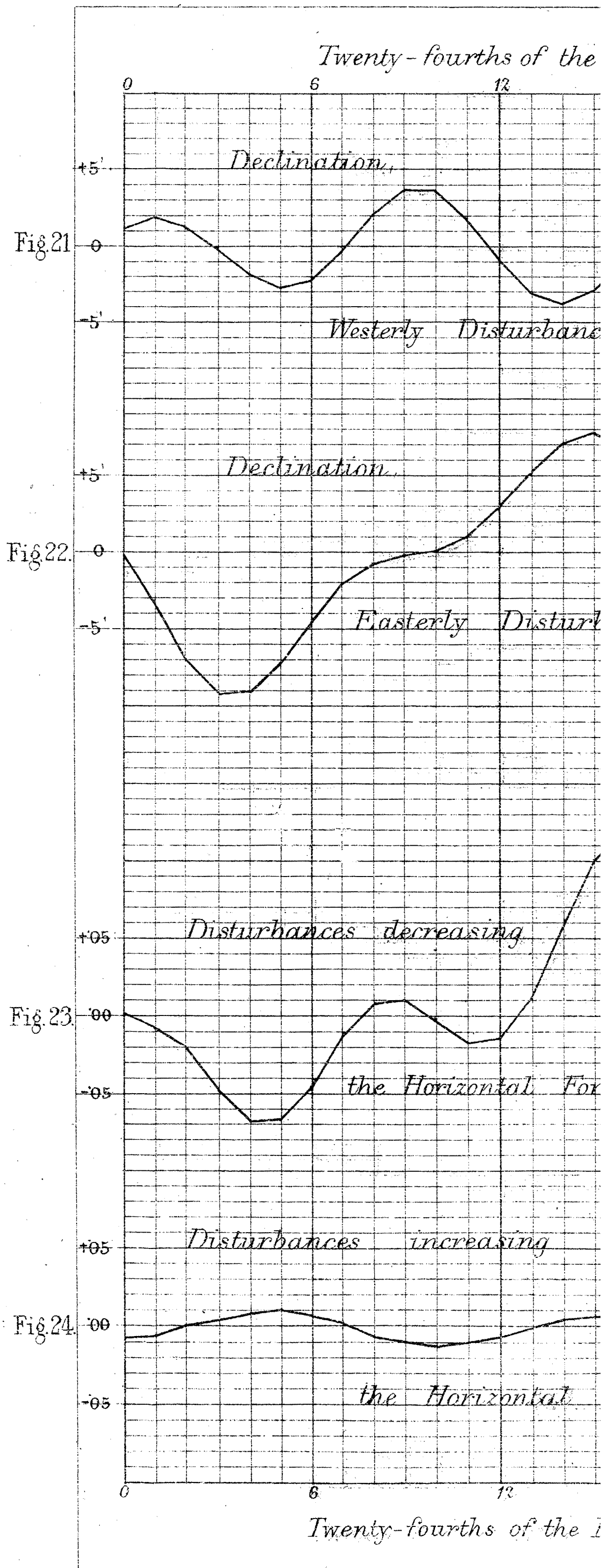
and Horizontal Force in the Synodic Periods of Mercury, Venus and Jup

VENUS.



cal thick lines mark the time of inferior Conjuntion.

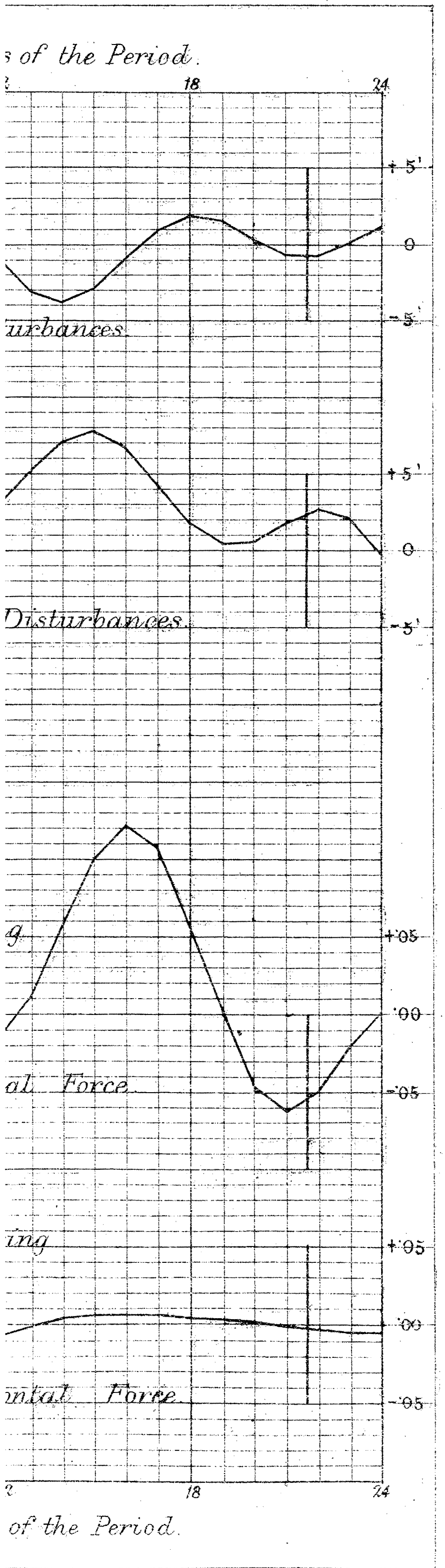
JUPITER



The vertical thick lines mark the

Jupiter.

TER.



rk the time of Opposition.

The vertical thick lines mark the time of inferior Conjuntion.

The vertical thi

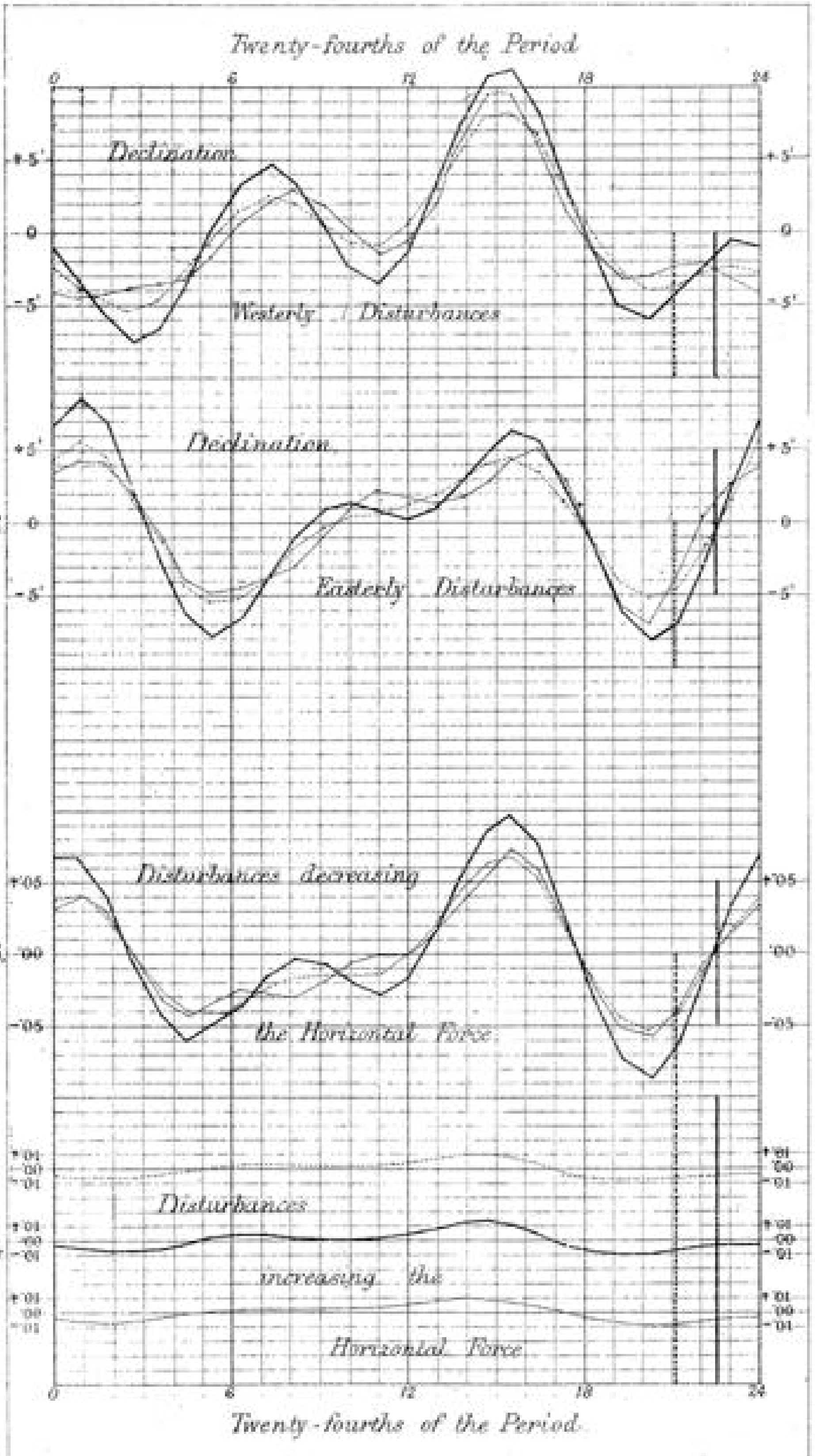
W.H.Wesley lith.

cal thick lines mark the time of inferior Conjuntion.	The vertical thick lines mark the *rk the time of Opposition.*

W. West & C° lith.

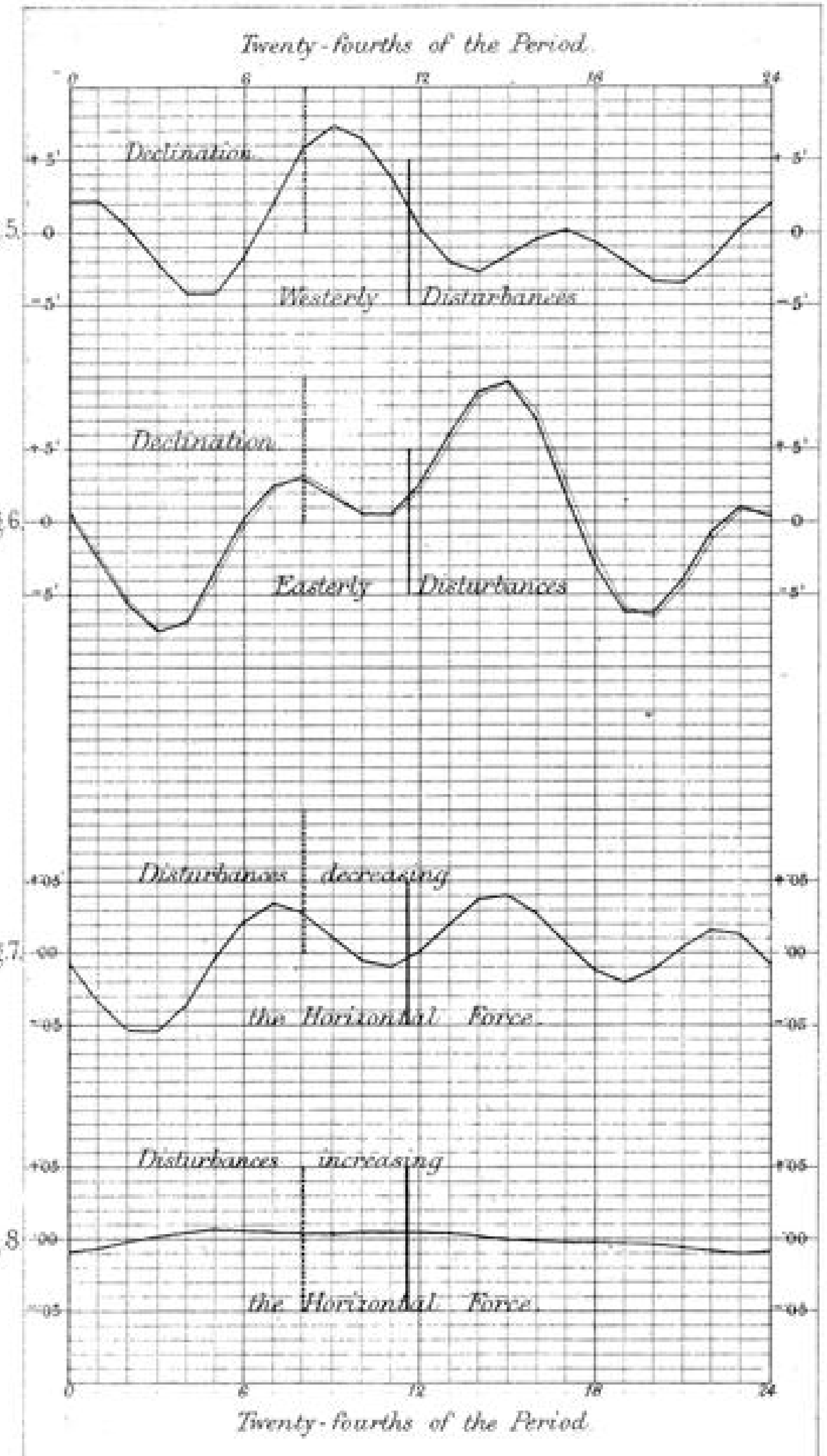
Disturbance Variations of Magnetic Declination and Horizontal Force in the Sidereal Periods of Mercury, Venus and the Earth.

MERCURY.



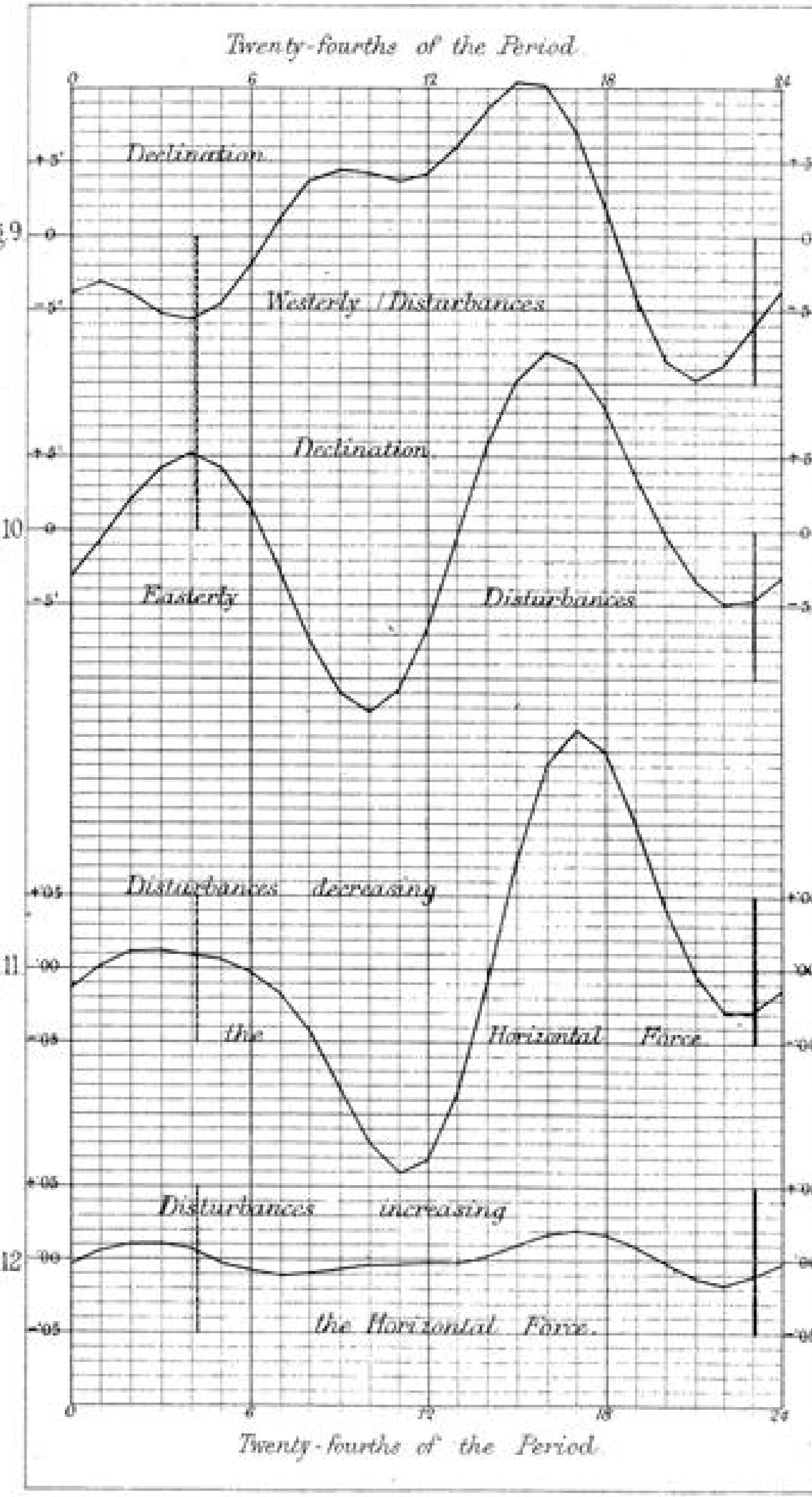
The vertical dotted and thick lines mark the times of the ascending Node and Perihelion respectively.

VENUS.



The vertical dotted and thick lines mark the times of the ascending Node and Perihelion respectively.

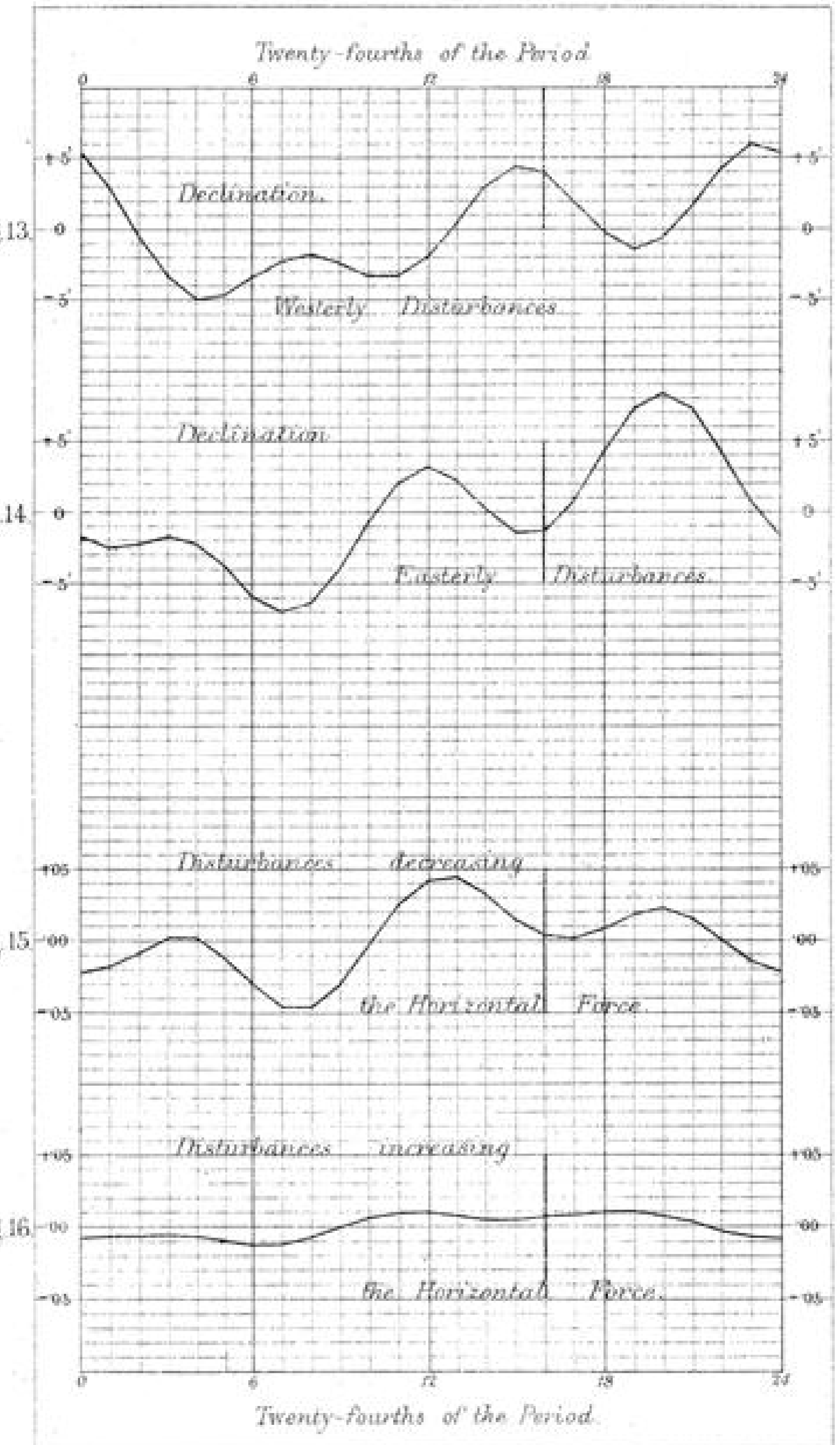
THE EARTH.



The Vertical dotted lines mark the time of the Vernal Equinox, and the Vertical thick lines the time of Perihelion.

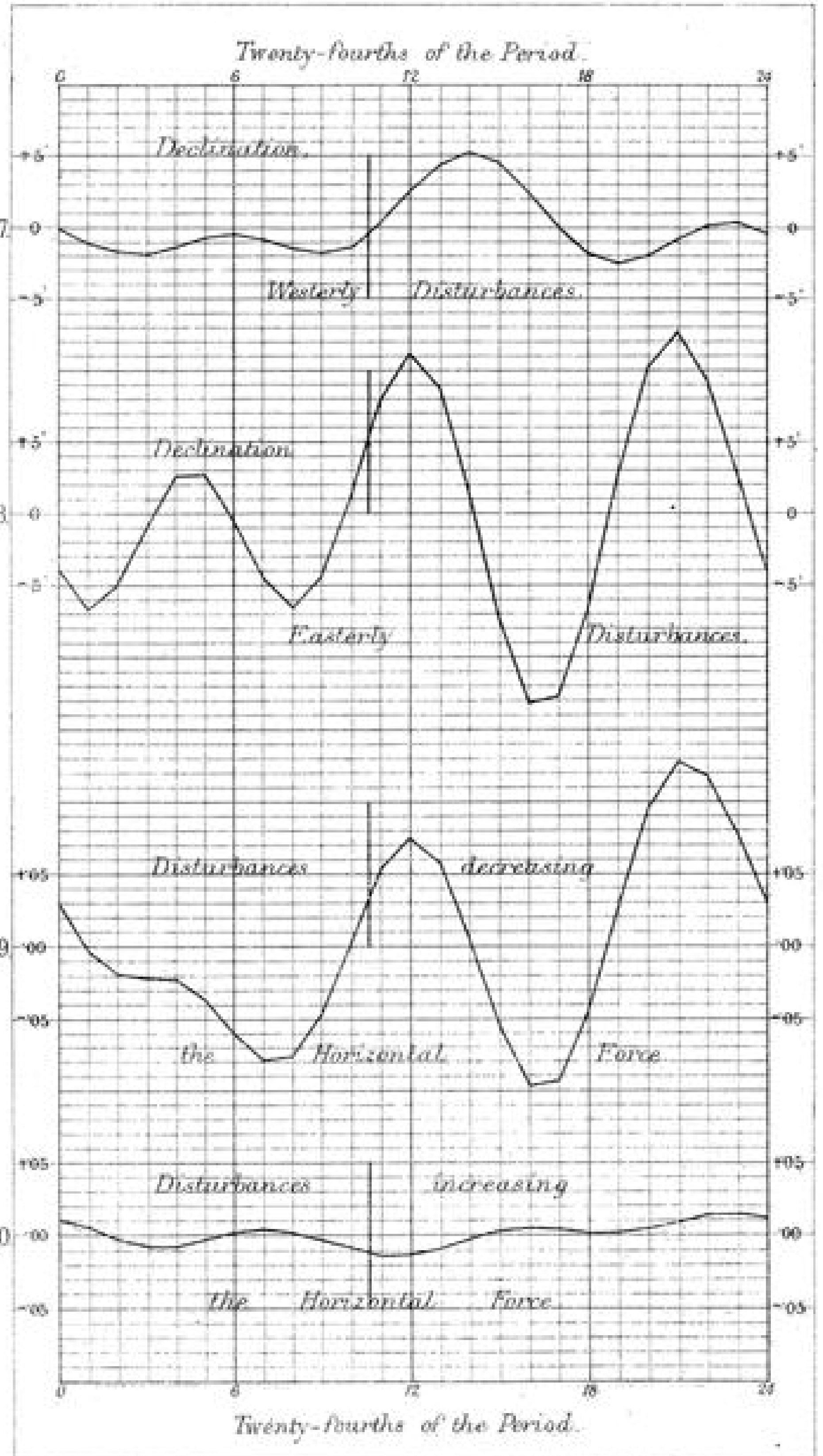
Disturbance Variations of Magnetic Declination and Horizontal Force in the Synodic Periods of Mercury, Venus and Jupiter.

MERCURY.



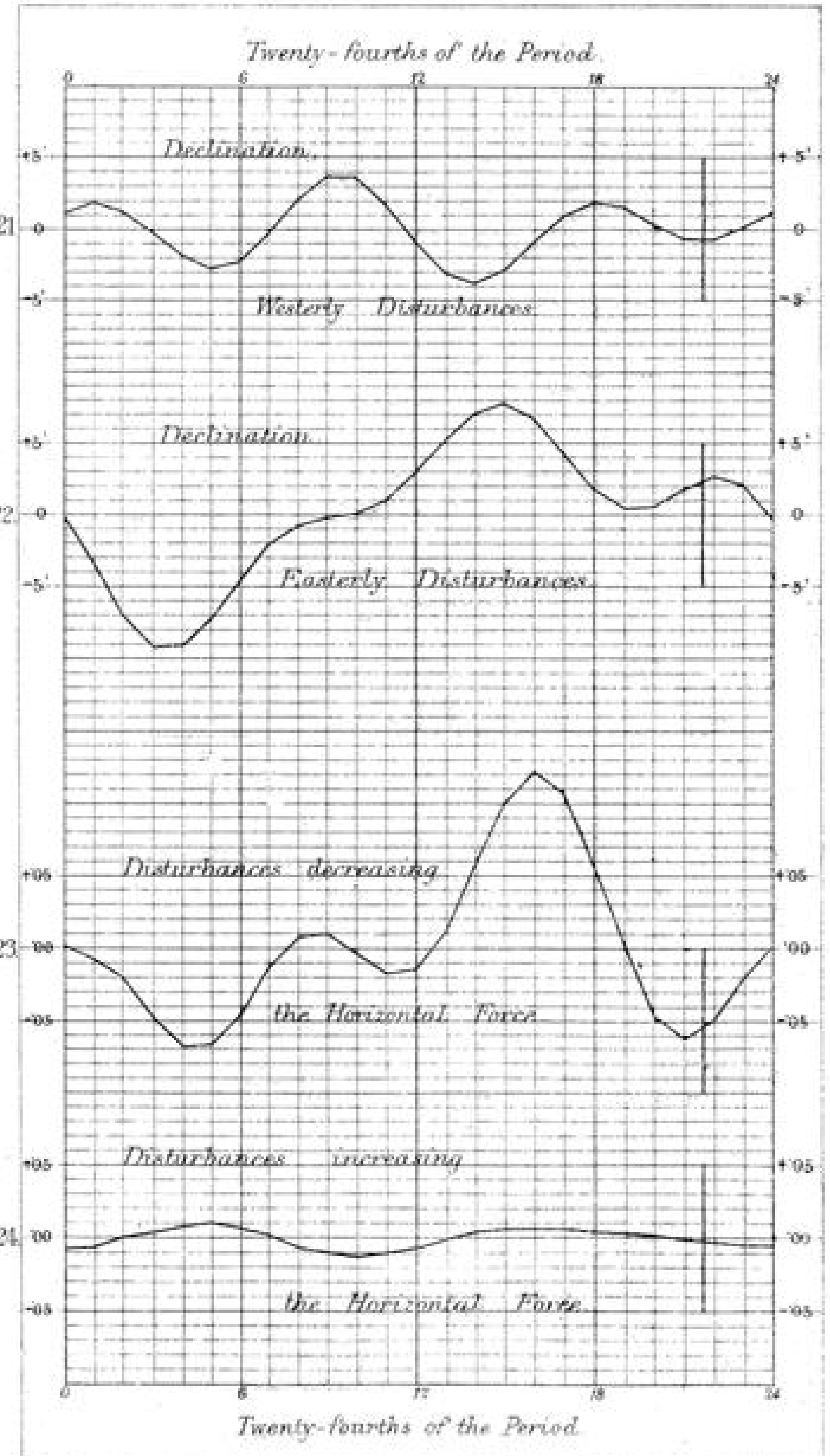
The vertical thick lines mark the time of inferior Conjuntion.

VENUS.



The vertical thick lines mark the time of inferior Conjuntion.

JUPITER.



The vertical thick lines mark the time of Opposition.